HOMEOMORPHISMS ON A SOLID TORUS

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1. **Introduction.** An orientable 3-manifold with boundary can be represented as a solid torus H ("cube with handles") plus a disjoint collection of 3-cells attached to the boundary of H along annuli.[1] Hence, a natural (but difficult) approach to the study of such 3-manifolds is to associate with each a system of disjoint simple closed curves in the boundary of a solid torus. It would be useful (e.g., in proving that two 3-manifolds are homeomorphic) to have conditions under which one such system of curves is topologically equivalent to another.

A special case of this problem is considered here. Let H be a solid torus of genus n and let J and J^* be two collections of n disjoint simple closed curves in the boundary of H. The result is that if each of the collections "generates" $\pi_1(H)$ (see §2), then the collections are topologically equivalent. Some care must be exercised in producing the homeomorphism, since not every homeomorphism on the boundary of H which throws one collection onto the other can be extended to H.

2. **Preliminaries.** Two simplicial complexes will be called *equivalent* if they have rectilinear subdivisions which are isomorphic complexes. An *n-cell* (*n-sphere*) is a complex equivalent to an n-simplex (boundary of an $n+1$-simplex, respectively). If the closed star of each vertex in the complex M is equivalent to an n-cell, then M is by definition an *n-manifold*. The union of those simplexes of M each of whose links is not a sphere is the *boundary* of M (Bd M) and the *interior* of M (Int M) is $M-\text{Bd } M$. All the manifolds considered here are to be compact. A compact manifold with null boundary is *closed*.

A continuous mapping between complexes is *piecewise linear* if it is simplicial relative to some rectilinear subdivision of each. A closed subset P of a complex K is *polyhedral* if the inclusion map $i: P \to K$ is piecewise linear for some triangulation of P. All mappings employed here are understood to be piecewise linear unless stated otherwise ("piecewise linear" is sometimes used explicitly for emphasis). *Regu-*

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[1] To see this, note that since the 3-manifold M has nonempty boundary, it contracts [4] into a 2-complex K which lies in the 2-skeleton of M. A "small" regular neighborhood of K in M clearly is of the form described above and, by Theorem 23 of [4], is equivalent to M. In case M is contractible, we may (as in our Theorem) take the number of annuli equal to the genus of H.

lar neighborhoods of subcomplexes of manifolds are understood in the sense of Whitehead [4].

A *solid torus* of genus $n \geqq 0$ is a 3-manifold M equivalent to the regular neighborhood in S^3 of a finite, connected 1-dimensional complex with first Betti number n. The boundary of M is a closed orientable surface of genus n. We write $g(M)=n$.

If f and g are mappings of $I=[0, 1]$ into a space X such that $f(1)=g(0)$, let $f \times g$ denote the mapping of I into X given by $(f \times g)(t) = f(2t)$ for $0 \leqq t \leqq 1/2$ and $(f \times g)(t) = g(2t-1)$ for $1/2 \leqq t \leqq 1$. Let $\bar{f}$ denote the mapping given by $\bar{f}(t)=f(1-t)$ for $0 \leqq t \leqq 1$. If $f(0)=f(1)=p$, then $[f]$ is the element of $\pi_1(X, p)$ determined by f.

Let H be a solid torus of genus n and $J_1 \cdots J_n$ disjoint polyhedral simple closed curves on Bd H. Select piecewise linear mappings $f_1, \cdots, f_n$, where $f_i: I \to J_i$, $f_i(0)=f_i(1)$, and $f_i|(0, 1)$ is a homeomorphism (the property to follow will not depend upon this selection). If there exists a point p in Bd H and piecewise linear mappings $\sigma_1, \cdots, \sigma_n$ where $\sigma_i: I \to \mathrm{Bd}\, H$ such that $\sigma_i(0)=p$, $\sigma_i^{-1}(J_i)=\{1\}$, $\sigma_i(1)=f_i(0)=f_i(1)$, $\sigma_i(I) \cap J_j = \emptyset$ for $i \neq j$, and the elements $[(\sigma_1 \times f_1) \times \bar{\sigma}_1], \cdots, [(\sigma_n \times f_n) \times \bar{\sigma}_n]$, generate $\pi_1(H, p)$, then it will be said that $J_1, \cdots, J_n$, *generate* $\pi_1(H)$ or form a *set of generators* for $\pi_1(H)$. Two sets of generators are *equivalent* if there is a piecewise linear homeomorphism of H onto H throwing the elements of one collection onto those of the other.

The following result seems to be generally known, and is an easy consequence of the Loop Theorem [1] and Dehn's Lemma [2].

LEMMA. *Let M be a compact 3-manifold with connected boundary of genus n. If M can be embedded in E^3 and $\pi_1(M)$ is a free group, then M is a solid torus and $g(M)=n$.*

3. **The Theorem.** We show the following:

THEOREM. *Let H be a solid torus with $g(H)=n$. Then, any two sets of generators for $\pi_1(H)$ are equivalent.*

PROOF. Let $J_1, \cdots, J_n$ generate $\pi_1(H)$. We show first that, if $n \geqq 2$, there is a polyhedral 2-cell D in H such that $D \cap \mathrm{Bd}\, H = \mathrm{Bd}\, D$, $\bigcup_{i=1}^n J_i \cap D = \emptyset$, and D separates the J_i's nontrivially in H. The remainder of the argument will then be an easy induction.

To do this, let ρ be a combinatorial isomorphism of H onto H^*, a disjoint copy. Let $N(J_1, \mathrm{Bd}\, H), \cdots, N(J_n, \mathrm{Bd}\, H)$ be disjoint regular neighborhoods in Bd H of the J_i's. Form the compact 3-manifold M by identifying points which correspond under

$\rho|\operatorname{Bd} H-\bigcup_{i=1}^{n}\operatorname{Int} N(J_i, \operatorname{Bd} H)=S'$. Let $\eta: H\cup H^*\to M$ be the identification map, and let $S=\eta(S')$.

Note that Bd M consists of closed surfaces (2-manifolds) of genus one: $T_1, \cdots, T_n$. Since $\pi_1(H)$ is free of rank n and is generated by n loops in S', it follows from van Kampen's Theorem [3] that $\pi_1(M)$ is freely generated by n loops in S. For future reference, notice that each of these loops determines a conjugate class of elements in $\pi_1(M)$ which coincides with exactly one of the classes determined by $\eta(J_1), \cdots, \eta(J_n)$. Hence M could be embedded in a simply-connected 3-manifold by attaching one solid torus to each T_i in such a way that each $\eta(J_i)$ becomes null-homotopic. Thus each closed surface in Int M separates M.

Consider the inclusion homomorphism $i^*: \pi_1(T_1)\to\pi_1(M)$. Since $\pi_1(M)$ is a free group and $\pi_1(T_1)$ is not, the kernel of i^* is nontrivial. By [1] and [2], there is a polyhedral 2-cell Δ in M such that $\Delta\cap\operatorname{Bd} M=\Delta\cap T_1=\operatorname{Bd}\Delta$ and Bd Δ does not separate T_1. A regular neighborhood in M of $\Delta\cup T_1$ has a 2-sphere K' as one boundary component and K' lies in Int M and separates T_1 from $\bigcup_{i=2}^{n} T_i$. Thus, there is a polyhedral 2-sphere K' which (1) lies in Int M, (2) is in general position relative to S, and (3) separates the T_i's nontrivially in M. Let K be a 2-sphere with these three properties and with the number m of components of $K\cap S$ (these are simple closed curves) minimal. By (3), $m\neq 0$.

No component of $K\cap S$ can bound a 2-cell on $\eta(\operatorname{Bd} H)$ or on $\eta(\operatorname{Bd} H^*)$. If this were to occur, there would be a 2-cell $E\subseteq S$ such that Bd $E\subseteq K$ and $K\cap\operatorname{Int} E=\varnothing$. Then by splitting $K\cup E$ along E one obtains 2-spheres K_0, K_1, lying in the union of K with a small neighborhood of E, each having properties (1) and (2) above but meeting S in fewer than m simple closed curves. Hence K_0 and K_1 must fail to have property (3), which implies that K does not have property (3), a contradiction.

Now we show how to choose the required D. Let D' be a polyhedral 2-cell in K such that Bd $D'\subseteq S$ and $S\cap\operatorname{Int} D'=\varnothing$. Since $S\cap\operatorname{Int} D'=\varnothing$, assume (say) that Int $D'\subseteq\eta(\operatorname{Int} H)$. We show that D' separates the $\eta(J_i)$'s nontrivially in $\eta(H)$, so that $\eta^{-1}(D')\cap H$ can be taken as D.

To do this, construct a 3-manifold M^* containing $\eta(H)$ by attaching 3-cells to $\eta(H)$ along each annulus $\eta(N(J_i, \operatorname{Bd} H))$. van Kampen's Theorem and the fact that the $\eta(J_i)$'s generate the fundamental group of $\eta(H)$ imply that $\pi_1(M^*)$ is trivial and hence Bd M^* is a 2-sphere. Since $D'\cap\operatorname{Bd} M^*=\operatorname{Bd} D'$, D' separates M^* and since

$D' \subseteq \eta(H)$, D' separates $\eta(H)$ also. If the $\eta(J_i)$'s were separated trivially in $\eta(H)$ by D', then Bd D' would bound a 2-cell on $\eta(\text{Bd } H)$, in contradiction to a previous observation. This completes the proof of the first assertion.

The proof of the Theorem now proceeds by an easy inductive verification of:

P(n): *Let H and H^* be solid tori, $g(H)=n=g(H^*)$, and let $J_1, \cdots, J_n$ and $J_1^*, \cdots, J_n^*$ be polyhedral simple closed curves which generate $\pi_1(H)$ and $\pi_1(H^*)$, respectively. Then there is a piecewise linear homeomorphism of H onto H^* which throws the elements of the first collection onto those of the second.*

P(1) is known. We prove P(n) for $n \geqq 2$ assuming P(i) for $1 \leqq i < n$. We assume without loss of generality that $J_1^*, \cdots, J_n^*$ are curves "nicely" located on a standard solid torus H^* of genus n in E^3. As shown above, there is a polyhedral 2-cell D such that $D \cap \text{Bd } H = \text{Bd } D$, $D \cap \bigcup_{i=1}^n J_i = \emptyset$, and D separates the J_i's nontrivially in H.

By the Lemma, $H = H_0 \cup H_1$ where H_0, H_1 are solid tori and $H_0 \cap H_1 = D$. Assume the notation chosen so that the first k of the J_i's lie in H_0 and the remaining $(n-k)$ in H_1 $(1 \leqq k \leqq n-1)$. Because we dictated the location of the J_i^*'s on H^*, we can write $H^* = H_0^* \cup H_1^*$ with $H_0^* \cap H_1^* = D^*$ (D^* is a 2-cell and H_i^* is a solid torus), the first k of the J_i^*'s in H_0^*, and the remainder in H_1^*. Also $g(H_0^*) = k$ and $g(H_1^*) = n-k$.

Since H_0, H_1 are retracts of H, $J_1, \cdots, J_k$ generate $\pi_1(H_0)$ and $J_{k+1}, \cdots, J_n$ generate $\pi_1(H_1)$. Since $g(H_0) \leqq k$, $g(H_1) \leqq n-k$ (the fundamental group of a solid torus is free of rank equal to its genus), and $g(H_0) + g(H_1) = n$, we have $g(H_0) = k$ and $g(H_1) = n-k$.

Let $N(D, H)$ and $N(D^*, H^*)$ be regular neighborhoods of D and D^* in H and H^*, respectively. These neighborhoods are 3-cells and are chosen to be disjoint from the generating curves and so that $N(D, H) \cap \text{Bd } H$ is a regular neighborhood of Bd D in Bd H, and similarly for $N(D^*, H^*)$. Denote by M_i and M_i^* $(i=0, 1)$ the closures of the components of $H - N(D, H)$ and $H^* - N(D^*, H^*)$, respectively, such that $M_i \subseteq H_i$ and $M_i^* \subseteq H_i^*$.

The pair (M_i, M_i^*) satisfies (with respect to the appropriate generating curves) the hypothesis of P(k) when $i=0$ and that of P($n-k$) when $i=1$ (since the same is true of (H_i, H_i^*)). Let h_0 be a homeomorphism of M_0 onto M_0^* taking $J_1, \cdots, J_k$ onto $J_1^*, \cdots, J_k^*$ in some order. Let h_1 be a homeomorphism of M_1 onto M_1^* taking $J_{k+1}, \cdots, J_n$ onto $J_{k+1}^*, \cdots, J_n^*$ in some order. We may now adjust h_0 so as to map the 2-cell $N(D, H) \cap M_0$ onto the 2-cell $N(D^*, H^*)$

$\cap M_0^*$ by following h_0 by an autohomeomorphism of M_0^* which throws $h_0(N(D, H)\cap M_0)$ onto $N(D^*, H^*)\cap M_0^*$ and is the identity on $J_1^*, \cdots, J_k^*$. Similarly for h_1.

We also wish to have h_0 and h_1 map the two boundary components of $N(D, H)\cap \mathrm{Bd}\, H$ onto the two boundary components of $N(D^*, H^*)\cap \mathrm{Bd}\, H^*$ *in the same sense.* If this is not the case, follow h_0 with another autohomeomorphism of M_0^* which reverses the orientation of the curve $M_0^*\cap N(D^*, H^*)\cap \mathrm{Bd}\, H^*$ and maps the curves $J_1^*, \cdots, J_k^*$ among themselves. With this done, we can find a homeomorphism h of $H-\mathrm{Int}\, N(D, H)$ onto $H^*-\mathrm{Int}\, N(D^*, H^*)$ which extends both h_0 and h_1. Finally, h extends to map $\mathrm{Int}\, N(D, H)$ onto $\mathrm{Int}\, N(D^*, H^*)$ homeomorphically and this completes the proof.

Added in proof. Essentially, the same result has been obtained (by different methods) by Heiner Zieschang in Abh. Math. Sem. Univ. Hamburg **25** (1962), 231–250 (see *Satz* 2, p. 239). The paper by Zieschang appeared after this paper was accepted. I wish to thank C. D. Papakyriakopoulos for bringing this to my attention.

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