MINIMAL REGULAR SPACES

MANUEL P. BERRI AND R. H. SORGENFREY

1. **Introduction.** If $\mathcal{P}$ is a property of topologies, a space $(X, \mathfrak{J})$ is minimal $\mathcal{P}$ if $\mathfrak{J}$ has property $\mathcal{P}$, but no topology on X which is strictly weaker (=smaller) than $\mathfrak{J}$ has $\mathcal{P}$. Such spaces have been investigated for the case $\mathcal{P}$=Hausdorff [**2**; **5**], a well-known result being that while every compact space is minimal Hausdorff, the converse is not true. We consider here the case $\mathcal{P}$=regular;[1] other properties are discussed by one of the authors in a paper to appear.

Filter-bases on spaces will be used extensively (for definitions not given here, see [**1**]). A filter-base is *open* (*closed*) if its elements are open (closed) sets. A filter-base will be called *regular* if it is open and is equivalent to a closed filter-base. The name is suggested by the fact that the filter-base of open neighborhoods of a point of a regular space is regular since it is equivalent to the filter-base of closed neighborhoods of that point.

2. **Characterizations of minimal regular spaces.** We will be concerned with spaces satisfying one or both of the following conditions:

(α) Every regular filter-base which has a unique adherent point is convergent.

(β) Every regular filter-base has an adherent point.

THEOREM 1. *A regular space which satisfies* (α) *also satisfies* (β).

PROOF. Suppose $\mathcal{B}$ is a regular filter-base on the regular space $(X, \mathfrak{J})$ and that $\mathcal{B}$ has no adherent point. Let $\mathcal{C}$ be a closed filter-base equivalent to $\mathcal{B}$. Fix $p \in X$ and let $\mathfrak{U}$ and $\mathfrak{V}$ be the filter-bases of open and closed neighborhoods of p, respectively. Since $\mathfrak{J}$ is regular, $\mathfrak{U}$ and $\mathfrak{V}$ are equivalent. Then $\mathcal{R} = \{B \cup U: B \in \mathcal{B}, U \in \mathfrak{U}\}$ is an open filter-base equivalent to the closed filter-base $\{C \cup V: C \in \mathcal{C}, V \in \mathfrak{V}\}$ and is therefore regular. It is clear that p is the unique adherent point of $\mathcal{R}$ and that $\mathcal{R}$ does not converge to p. This denial of the hypothesis establishes the theorem.

THEOREM 2. *In order that a regular space be minimal regular, it is necessary and sufficient that it satisfy* (α).

PROOF. Suppose $(X, \mathfrak{J})$ is regular and that $\mathcal{B}$ is a regular filter-base having the unique adherent point p to which it does not converge.

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[1] As used in this paper, the condition of regularity includes T_1 separation, i.e., singletons are closed.

For each $x \in X$, let $\mathfrak{U}(x)$ be the filter-base of $\mathfrak{I}$-open neighborhoods of x and define $\mathfrak{U}'(x) = \mathfrak{U}(x)$ if $x \neq p$ and $\mathfrak{U}'(p) = \{U \cup B: U \in \mathfrak{U}(x), B \in \mathfrak{B}\}$. There is a topology $\mathfrak{I}'$ on X such that $\mathfrak{U}'(x)$ is an open base at x for each $x \in X$. It is clear that $\mathfrak{I}'$ is strictly weaker than $\mathfrak{I}$ (there is a $U \in \mathfrak{U}(p)$ which contains no set of $\mathfrak{U}'(p)$ since $\mathfrak{B}$ does not converge to p). Moreover, $\mathfrak{I}'$ is certainly regular at each $x \neq p$, while regularity at p follows readily from the fact that $\mathfrak{B}$ is equivalent to a closed filter-base. Hence $\mathfrak{I}$ is not minimal regular.

To establish the sufficiency of the condition, let $(X, \mathfrak{I})$ be a regular space satisfying (α) and let $\mathfrak{I}'$ be a regular topology on X which is weaker than $\mathfrak{I}$. For arbitrary $x \in X$ let $\mathfrak{U}(x)$ and $\mathfrak{U}'(x)$ be the open neighborhood systems of x in the $\mathfrak{I}$ and $\mathfrak{I}'$ topologies, respectively. The filter-base $\mathfrak{U}'(x)$ is $\mathfrak{I}'$-regular and has x as its only adherent point. Since $\mathfrak{I}'$ is weaker than $\mathfrak{I}$, $\mathfrak{U}'(x)$ is regular and has unique adherent point x in $(X, \mathfrak{I})$. By (α) $\mathfrak{U}'(x)$ converges to x in $(X, \mathfrak{I})$. Hence $\mathfrak{U}(x)$ must be weaker than $\mathfrak{U}'(x)$, and, since the reverse is true, it follows that $\mathfrak{I}$ and $\mathfrak{I}'$ are identical and that $\mathfrak{I}$ is minimal regular.

REMARK. The two previous results show that condition (β) is necessary in order that a regular space be minimal regular. Whether it is sufficient is an open question. Theorem 3 below, however, throws some light on the problem.

LEMMA. *If the subspace X of the regular space Y satisfies (β), then X is closed in Y.*

PROOF. Suppose $p \in \overline{X} - X$. Let $\mathfrak{U}$ and $\mathfrak{V}$ be, respectively, the open and closed neighborhood systems of p in Y. Then the filter-base $\mathfrak{B} = \{X \cap U: U \in \mathfrak{U}\}$ is open (relative to X), is equivalent to the closed (relative to X) filter-base $\{X \cap V = V \in \mathfrak{V}\}$, and is therefore regular on X. As a filter-base on Y, $\mathfrak{B}$ is stronger than $\mathfrak{U}$ and hence has no adherent point other than p in Y. It follows that $\mathfrak{B}$ has no adherent point at all in X, a contradiction.

THEOREM 3. *Any completely regular space satisfying (β) is compact and therefore minimal regular.*

PROOF. Let X be completely regular and satisfy (β) and let Y be its Stone-Čech compactification. The above lemma yields the desired result.

THEOREM 4. *Any minimal regular subspace of a regular space is closed.*

PROOF. This is an immediate consequence of the lemma since the subspace must satisfy (β).

REMARK. It is easy to see that a subspace of a minimal regular space which is both open and closed is itself minimal regular. The example of the next section shows that a closed subspace of a minimal regular space need not be minimal regular.

3. **A minimal regular noncompact space.** The example given here is a slight modification of an unpublished one due to Richard Arens of a regular space which is not completely regular. His example has also been used by Hewitt [3] in constructing a regular space on which every continuous real-valued function is constant.

Description of the space $(Z, \mathfrak{J})$. Let J be the set of all integers, ω' the ordinals $\leqq\omega$, and Ω' the ordinals $\leqq\Omega$ (the first uncountable one). Equip each of these sets with the order topology and consider the space $J\times\omega'\times\Omega'-\{(n, \omega, \Omega): n\in J\}$, the relative product topology being used. To obtain the space Y, make the following identifications and use the quotient topology $\mathfrak{J}^*$: for even n, identify (n, ω, y) and $(n+1, \omega, y)$; for odd n, identify (n, x, Ω) and $(n+1, x, \Omega)$. We will continue to use the symbols (n, x, y) for the points of Y, thus $(n, \omega, y) = (n+1, \omega, y)$ for even n. For $n\in J$, let $Q_n=\{(n, x, y): x<\omega, y<\Omega\}$ and $Z_n=\overline{Q}_n=\{(n, x, y): (x, y) \neq(\omega, \Omega)\}$. Let p and q be points not in Y and topologize $Z=\{p\}\cup\{q\}\cup Y$ by letting an open base at p be all sets of the form

$$V_n(p) = \bigcup \{Z_i: i > n\} \cup Q_n \cup \{p\}, \qquad n = 1, 2, \cdots,$$

and an open base at q be all sets of the form

$$V_n(q) = \bigcup \{Z_{-i}: i > n\} \cup Q_{-n} \cup \{q\}, \qquad n = 1, 2, \cdots,$$

which open bases at points of Y are those they had in $\mathfrak{J}^*$. Let the resulting topology on Z be $\mathfrak{J}$.

Properties of the space $(Z, \mathfrak{J})$. 1. $(Z, \mathfrak{J})$ is regular.

PROOF. It is easy to see that singletons are closed, and regularity is clear except possibly at p and q. Regularity at p, say, follows from $[V_{n+1}(p)]^-\subset V_n(p)$.

We will say that a set $S\subset Z$ *gets into the n-corner* if whenever $x_0<\omega$, $y_0<\Omega$, there is a point $(n, x, y)\in S$ for some $x>x_0$ and $y>y_0$.

2. If the open set U gets into the n-corner, then there is an infinite sequence $\{x_i\}$ of distinct finite ordinals such that $(n, x_i, \Omega)\in\overline{U}$.

PROOF. If not, there is an $x_0<\omega$ such that if $x_0<x<\omega$, $(n, x, \Omega)\notin\overline{U}$ and hence there is a $y_x<\Omega$ such that $(n, x, y)\notin U$ for $y_x<y$. Since $\{y_x: x_0<x<\omega\}$ is countable, its least upper bound, y_0, is less than Ω. Therefore if $x_0<x<\omega$ and $y_0<y$, then $(n, x, y)\notin U$. Since U gets into the n-corner, it must then be that $(n, \omega, y)\in U$ for some $y>y_0$. But since U is open, there is an x, $x_0 < x < \omega$, such that $(n, x, y) \in U$. This contradiction establishes the property.

3. Let U, V, and W be open sets such that $U \subset \overline{U} \subset V \subset \overline{V} \subset W$. Then if U gets into the n-corner, W gets into the $(n-1)$- and $(n+1)$-corners.

Proof for n odd. (The proof for the case n even is similar.) Take $x_0 < \omega$, $y_0 < \Omega$. By property 2, there are infinitely many distinct x_i such that $x_0 < x_i < \omega$ and $(n, x_i, \Omega) \in \overline{U}$. Since $(n+1, x_i, \Omega) = (n, x_i, \Omega) \in \overline{U} \subset W$, W gets into the $(n+1)$-corner. Since $(n, x_i, \Omega) \in V$, there exists, for each i, a $y_i < \Omega$ such that if $y > y_i$, then $(n, x_i, y) \in V$. Let y' be the least upper bound of the set $\{y_0, y_1, y_2, \cdots\}$. Then for any y, $y' < y < \Omega$, $(n, x_i, y) \in V$ for all x_i; hence $(n, \omega, y) = (n-1, \omega, y) \in \overline{V} \subset W$, and W gets into the $(n-1)$-corner.

4. If $\mathfrak{B}$ is a regular filter-base and, for some n, each set of $\mathfrak{B}$ gets into the n-corner, then p and q are adherent points of $\mathfrak{B}$.

Proof. Let N be a neighborhood of p and $B \in \mathfrak{B}$. There is an integer k such that $Q_k \subset V_k(p) \subset N$; let $h = k - n$. Since $\mathfrak{B}$ is regular, there are $2h+1$ sets $U_i \in \mathfrak{B}$ such that

$$U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset \cdots \subset U_{2h+1} = B.$$

Since U_1 gets into the n-corner, h applications of property 3 shows that $B = U_{2h+1}$ gets into the $n+h=k$-corner; i.e., $B \cap Q_k \neq 0$, whence $B \cap N \neq 0$, and p is an adherent point of $\mathfrak{B}$. The case for q is similar.

5. $(Z, \mathfrak{I})$ is not completely regular and hence not compact.

Proof. Let f be a bounded, real-valued continuous function on Z. For some fixed n and each $y < \Omega$, let $g(y) = f(n, \omega, y)$. Then g is continuous, and it is well-known (e.g., [**4**, p. 167, ex. Q]) that there is a $y_0 < \Omega$ and a constant c such that $g(y) = c$ for $y > y_0$. It follows that each set of the regular filter-base $\{\{p \in Z: |f(p) - c| < \epsilon\} : \epsilon > 0\}$ gets into the n-corner. Since, by property 4, p and q are adherent points of this filter-base, it is clear that $f(p) = f(q) = c$ and $(Z, \mathfrak{I})$ is not completely regular.

In the proof of the following property we repeatedly use the elementary fact that if $\mathfrak{B}$ is a regular filter-base and $C \in \mathfrak{B}$, then $\mathfrak{C} = \{C \cap B : B \in \mathfrak{B}\}$ is a regular filter-base equivalent to $\mathfrak{B}$. We will call $\mathfrak{C}$ the *C-section* of $\mathfrak{B}$.

6. $(Z, \mathfrak{I})$ is minimal regular.

Proof. Let $\mathfrak{B}$ be a regular filter-base with unique adherent point r. We will show that $\mathfrak{B}$ converges to r; the property will then follow from Theorem 2.

Case 1. $r \neq p, q$. Then some set $C \in \mathfrak{B}$ meets only a finite number of Z_n's. Let $\mathfrak{C}$ be the C-section of $\mathfrak{B}$; then there is an integer k such that each set of $\mathcal{C}$ is a subset of $K=\bigcup\{z_n: |n| \leqq k\}$. It follows from property 4 that for each n, $|n| \leqq k$, there is a set $D_n \in \mathcal{C}$ which does not get into the n-corner. Let D be a set of $\mathcal{C}$ lying in $\bigcap\{D_n: |n| \leqq k\}$; then ordinals $x_0<\omega$, $y_0<\Omega$ exist such that D does not meet the open set $W=\{(n, x, y): x>x_0, y>y_0\}$. Hence $\mathfrak{D}$, the D-section of $\mathcal{C}$, is a filter-base equivalent to $\mathcal{B}$, and each of its sets lies in the compact subspace $K-W$ of Z. It is clear that $\mathfrak{D}$, and hence $\mathcal{B}$, must converge to their unique adherent point r.

Case 2. $r=p$. (The proof for the case $r=q$ is similar.) If $\mathcal{B}$ does not converge to p, there is a neighborhood $V_k(p)$ which contains no set of $\mathcal{B}$. Since q is not an adherent point of $\mathcal{B}$, there is an integer h and a set C of $\mathcal{B}$ such that $C \cap Z_n=\varnothing$ for $n<h$. It follows from property 4 that for each n, $h \leqq n \leqq k$, there is a set D_n in the C-section $\mathcal{C}$ of $\mathcal{B}$ which does not get into the n-corner. Let D be a set of $\mathcal{C}$ lying in $\bigcap\{D_n: h \leqq n \leqq k\}$; then ordinals $x_0<\omega$ and $y_0<\Omega$ exist such that D does not meet the set $W=\{(n, x, y): h \leqq n \leqq k, x>x_0, y>y_0\}$. The D-section $\mathfrak{D}$ of $\mathcal{C}$ is a filter-base equivalent to $\mathcal{B}$ and each of its sets meets the compact set $F=\bigcup\{Z_n: h \leqq n \leqq k\}-W$. Hence $\mathcal{E}=\{F \cap E: E \in \mathfrak{D}\}$ is a filter-base stronger than $\mathcal{B}$ and each of its sets is contained in F. Since F is compact, $\mathcal{E}$, and hence $\mathcal{B}$, must have an adherent point $z \in F$. Since $z \neq p$, a contradiction results.

7. $(Z, \mathfrak{J})$ has a closed subspace which is not minimal regular.

PROOF. Let $S=\{(1, x, \Omega): x<\omega\}$. It is clear that S is a closed subset of Z. But, with the relative topology, S is an infinite discrete space, which is certainly not minimal regular.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES