ON THE BOUNDARY VALUES OF BLASCHKE PRODUCTS AND THEIR QUOTIENTS

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1. Introduction. Let \( B(z) \) be the infinite Blaschke product:

\[
e^{i\lambda} z^m \prod_{n=1}^{+\infty} \frac{\bar{a}_n (a_n - z)}{|a_n| (1 - \bar{a}_n z)},
\]

where \( \lambda \) is a real constant, and \( m \) is a non-negative integer, \( 0 < |a_n| < 1, \sum_{n=1}^{+\infty} (1 - |a_n|) < +\infty \). The object of this note is to establish the following two theorems.

**Theorem 1.** (A) If the subsequence \( \{a_{n_k}\} \) tends to \( z = e^{i\phi} \) within the Stolz domain in such a manner that

\[
\lim_{k \to +\infty} \frac{|a_{n_k} - a_{n_{k+1}}|}{|a_{n_k} - e^{i\phi}|} = 0,
\]

then the angular limit at \( e^{i\phi} \) of \( B(z) \) is 0.

(B) If the subsequence \( \{a_{n_k}\} \) tends to \( z = e^{i\phi} \) within the circle:

\[
|z - ae^{i\phi}| \leq 1 - a \quad (0 < a < 1),
\]

in such a manner that

\[
\lim_{k \to +\infty} 1/x_k^2 \cdot |a_{n_k} - a_{n_{k+1}}| = 0,
\]

where \( x_k = \min\{|a_{n_k} - e^{i\phi}|, |a_{n_{k+1}} - e^{i\phi}|\} \), then the angular limit at \( e^{i\phi} \) of \( B(z) \) is 0.

As an application of Theorem 1 (A), we prove

**Theorem 2.** There exists a meromorphic function \( f(z) \) of bounded characteristic in \( |z| < 1 \) represented by the quotient of two infinite Blaschke products such that

1. \( f(z) \) has infinite number of zeros and poles on \( \arg(1 - z) = -\theta \) and \( \arg(1 - z) = +\theta \) respectively \( (0 < \theta < \pi/2) \).
2. \( \lim_{z \to 1; \arg(1 - z) = -\theta} f(z) = 0 \) and \( \lim_{z \to 1; \arg(1 - z) = +\theta} f(z) = \infty \).

**Remark.** (1) O. Frostman [1, p. 109] was the first to construct an example of Blaschke product with the boundary value 0, i.e.,

\[
B(z) = \prod_{n=1}^{+\infty} \frac{(1 - 1/n^2) - z}{1 - (1 - 1/n^2)z}, \quad \text{where} \quad \lim_{r \to 1} B(r) = 0.
\]
By the well-known Iversen-Lindelöf theorem on asymptotic values, \( f(z) \) of Theorem 2 has Picard’s property in the sector \( S: |\arg(1-z)| \leq \theta < \pi/2; w=f(z) \) takes every value \( w \), except perhaps two, infinitely many times in \( S \). On the other hand, \( f(e^{i\theta}) \) is of modulus one almost everywhere on \( |z|=1 \).

D. A. Storvick [3, p. 37] constructed a meromorphic function \( f(z) \) defined by the quotient of two infinite Blaschke products such that \( f(z) \to 0 \) and \( \infty \) as \( z \to 1 \) along the upper and lower oricycle: \( r = \cos \theta \) respectively, \( z = re^{i\theta} \).

2. Proof of Theorem 1. (A) We decompose \( B(z) \) as follows:

\[
B(z) = B_1(z) \cdot B_2(z),
\]

where \( B_1(z) = \prod_{k=1}^{+\infty} \frac{\bar{a}_n(a_n - z)}{|a_n|} (1 - \bar{a}_n z), \quad B_2(z) = B(z)/B_1(z). \) Since \( |B(z)| < |B_1(z)| \) for \( |z| < 1 \), it is sufficient to prove that the angular limit at \( e^{i\theta} \) of \( B_1(z) \) is 0.

Without any loss of generality, we can assume that \( \phi = 0 \). Put \( z = 1 - re^{i\theta}, \quad a_n = b_1 = 1 - r_k e^{i\theta_k} \). By a simple calculation,

\[
(2.1) \quad \frac{(b_k - z)/(1 - \bar{b}_k z)}{(b_k - z)/(1 - \bar{b}_k z)} = \frac{(e^{i\theta_k} + 1) - re^{i\theta} + (b_k - z)/r_k e^{i\theta_k} \cdot e^{i\theta_k}}{(1 - b_k)}. \]

Let us denote by \( l_k \) the segment connecting two points \( b_k \) and \( b_{k+1} \). If \( z \) lies on \( l_k \), we have evidently

\[
(2.2) \quad |b_k - z| \leq |b_k - b_{k+1}|, \quad r \leq \max(r_k, r_{k+1}).
\]

By \( |e^{i\theta_k} + 1| > \sin(2\theta) \), we get easily

\[
(2.3) \quad |e^{i\theta_k} + 1| > \sin(2\theta), \quad 1/2.
\]

By (2.1), (2.2) and (2.3)

\[
|\frac{(b_k - z)/(1 - \bar{b}_k z)}{1 - \bar{b}_k z}| \leq |(b_k - b_{k+1})|/|1 - b_k| \cdot |\sin(2\theta) - \max(r_k, r_{k+1}) - |(b_k - b_{k+1})|/|1 - b_k| \}^{-1},
\]

so that, by the assumptions:

\[
\lim_{k \to \infty} |(b_k - b_{k+1})|/|1 - b_k| = 0, \quad \lim_{k \to \infty} \max(r_k, r_{k+1}) = 0,
\]

we obtain

\[
(2.4) \quad \lim_{k \to \infty} (b_k - z)/(1 - \bar{b}_k z) = 0,
\]

where \( z \in l_k \). Since

\[
|e^{i\theta_k} + 1| = 2 \cos \theta_k \geq 2 \cos \theta > \sin(2\theta).
\]
\[ |B_i(z)| < \left| \frac{(b_k - z)}{(1 - \bar{b}_k z)} \right| \quad \text{for any } k \text{ and } |z| < 1, \]

by (2.4)

\[ \lim_{z \to 1} B_i(z) = 0, \]
as \(z \to 1\) along \(C = \bigcup_k l_k\). Hence, by Lindelöf's theorem \([2, p. 5]\)

\[ \lim_{z \to 1} B_i(z) = 0, \]
as \(z \to 1\) inside a Stolz domain with vertex at \(z = 1\), as was to be proved.

B) Using the same notations as above, we get

\[ (b_k - z)/(1 - \bar{b}_k z) = (b_k - z)/rr_k e^{i(\theta - 8_k)} \cdot \left\{ e^{i\theta_k}/r_k + e^{-i\theta}/r - 1 \right\}^{-1}. \]

In the circle: \(|z - a| \leq 1 - a\) \((0 < a < 1, \ z = 1 - re^{i\theta})\), we have

\[ \frac{1}{2(1 - a)} \leq \cos \theta/r. \]

If \(z\) lies on \(l_k\), by (2.5) and (2.6),

\[ \left| \frac{(b_k - z)}{(1 - \bar{b}_k z)} \right| \leq \left| \frac{(b_k - z)}{rr_k \cdot \left\{ \cos \theta_k/r_k + \cos \theta/r - 1 \right\}^{-1}} \right| \leq \left( \frac{1}{a - 1} \right) \cdot \frac{y_k}{(\min(r) \cdot x_k)}, \]

where \(y_k = |b_k - b_{k+1}|, \ x_k = \min(r_s, r_{k+1}), \ \min(r) = \min_{z \in l_k} |z - 1|\). If \(\min(r) = x_k\), we have

\[ \left| \frac{(b_k - z)}{(1 - \bar{b}_k z)} \right| \leq \left( \frac{1}{a - 1} \right) \cdot \frac{y_k}{x_k^2}. \]

If \(\min(r) < x_k\), we have easily

\[ \min(r) \geq \left( x_k^2 - \left( \frac{y_k}{2} \right)^2 \right)^{1/2}, \]

so that

\[ \left| \frac{(b_k - z)}{(1 - \bar{b}_k z)} \right| \leq \left( \frac{1}{a - 1} \right) \cdot \frac{y_k}{x_k^2} \cdot \left\{ 1 - \left( \frac{y_k}{2x_k} \right)^2 \right\}^{-1/2}. \]

In any case, by the assumption: \(\lim_{k \to \infty} y_k/x_k^2 = 0\), we have

\[ \lim_{k \to \infty} \frac{(b_k - z)}{(1 - \bar{b}_k z)} = 0, \]
as \(z\) on \(l_k\). Hence, by entirely similar arguments as in (A),

\[ \lim_{z \to 1} B(z) = 0, \]
as \(z \to 1\) inside the Stolz domain with vertex at \(z = 1\).

3. Lemmas. To prove Theorem 2, we need two lemmas.

Lemma 1. Put
where $|a| < 1$, $I(a) > 0$. Then

$$|w(z)| < 1 \quad \text{for} \quad |z| < 1, \quad I(z) > 0.$$  

**Proof.** $w(z)$ is regular in the upper semi-circle $D: |z| \leq 1$, $I(z) \geq 0$. On the boundary of $D$, we have evidently $|w| = 1$. Hence, by the maximum-modulus principle, $|w(z)| < 1$ for $|z| < 1$, $I(z) > 0$.

**Lemma 2.** In the domain $D: |z| < 1$, $I(z) \geq 0$, $|z-1| \leq |a-1|$, where $|a| < 1$, $I(a) > 0$, we have

$$| (1 - az)/(z - a) | < \exp(2/\sin^2 \theta),$$

where $\arg(1-a) = -\vartheta$ ($0 < \vartheta < \pi/2$).

**Proof.** By the inequality: $\log(1+x) \leq x$ for $x \geq 0$, for $|a| < 1$, $|z| \leq 1$ we obtain

$$\log | (1 - az)/(z - a) | = \frac{1}{2} \log \left[ 1 + (1 - |a|^2)(1 - |z|^2)/|z-a|^2 \right] \leq \frac{1}{2} \frac{1}{|z-a|^2} \leq 2|1-a|^2/|1-a|^2.$$  

Hence

$$\log | (1 - az)/(z - a) | < 2|1-a|^2/|I(a)|^2 = 2/\sin^2 \theta \quad \text{for} \quad z \in D,$$

because $|z-1| \leq |a-1|$, $|z-a| \geq I(a)$ in $D$. Thus Lemma 2 is proved.

**4. Proof of Theorem 2.** Let the sequence $\{\epsilon_n\}$ be such that

$$\cos \vartheta > \epsilon_1 > \epsilon_2 > \cdots > \epsilon_n > \rightarrow 0,$$

(4.1)  

$$\sum_{n=1}^{\infty} \epsilon_n < + \infty,$$

$$\lim_{n \rightarrow \infty} \frac{\epsilon_{n+1}}{\epsilon_n} = 1.$$  

Put $a_n = 1 - \epsilon_n \cdot e^{-i\vartheta}$ ($0 < \vartheta < \pi/2$). Then

$$|a_n| < 1, \quad I(a_n) > 0 \quad \text{for} \quad n \geq 1.$$  

The desired function $f(z)$ is given by $f(z) = B_1(z)/B_2(z)$, where

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* $I(a)$ is the imaginary part of $a$.  

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\[ B_1(z) = \prod_{n=1}^{+\infty} \frac{a_n(a_n - z)}{a_n(1 - a_n z)}, \]
\[ B_2(z) = \prod_{n=1}^{+\infty} a_n(\bar{a}_n - z)/|a_n| (1 - a_n \bar{z}). \]

Since \( \sum_{n=1}^{+\infty} |1 - a_n| < \sum_{n=1}^{+\infty} |1 - a_n| = \sum_{n=1}^{+\infty} \epsilon_n < +\infty \), the Blaschke products \( B_i(z) \) \((i=1, 2)\) are convergent.

We can put
\[ f(z) = \prod_{n=1}^{+\infty} \frac{a_n/a_n \cdot (a_n - z)(1 - a_n z)/(1 - a_n \bar{z})(\bar{a}_n - z)}, \]
so that, by Lemmas 1 and 2, we have
\[ |f(z)| < |(a_k - z)/(1 - \bar{a}_k z)| \cdot \exp(2/\sin^2 \vartheta) \]
on the segment: \( \arg(1 - z) = -\theta, \ |1 - z| \leq \epsilon_k \). By (4.1)
\[ |(a_k - a_{k+1})/(1 - a_k)| = 1 - \epsilon_{k+1}/\epsilon_k \to 0 \quad \text{as} \ k \to +\infty. \]

Hence, by (4.2) and arguments similar to those in the proof of Theorem 1 (A),
\[ \lim_{z \to 1^+; \arg(1 - z) = -\theta} f(z) = 0. \]

Similarly
\[ \lim_{z \to 1^-; \arg(1 - z) = +\theta} 1/f(z) = 0. \]

Thus Theorem 2 is completely established.

REFERENCES


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