ON THE VALUE OF DETERMINANTS

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The following problems were suggested, in the January 1962 issue of the Bulletin:

What are the maximum values of nth order determinants subject to the conditions

(a) each element, $a_{rs}=0$ or 1,

(b) each element, $a_{rs}=-1$ or 1,

(c) each element, $a_{rs}=-1$, 0 or 1?

We shall not solve these problems completely, but we shall show that the three problems are equivalent and obtain the values approximately for large n.

Notation. Define $f(n)$, $g(n)$, $h(n)$ to be the maximum values of nth order determinants with elements subject to (a), (b), and (c) respectively and let F_n, G_n, H_n be the matrices satisfying the conditions whose determinants have values $f(n)$, $g(n)$ and $h(n)$. Of course these matrices are not unique.

Preliminaries.

THEOREM 1. *$g(n)=h(n)$ for each n.*

Certainly, since the class of matrices with elements -1, 0 or 1 contains the class with elements -1 or 1 therefore, $h(n)\geqq g(n)$.

Secondly, consider H_n. If H_n has no zero element then clearly $g(n)=h(n)$. If H_n has at least one zero element, suppose $a_{rs}=0$. Then consider the expansion by the rth row of $h(n)$. $h(n)=a_{r1}A_{r1}+a_{r2}A_{r2}+\cdots+a_{rn}A_{rn}$. If $A_{rs}>0$, we could increase $h(n)$ by replacing a_{rs} by 1. If $A_{rs}<0$ we could increase $h(n)$ by replacing a_{rs} by -1. If $A_{rs}=0$ we could replace a_{rs} by 1 without altering $h(n)$. Hence we may in turn replace each zero element of H_n without decreasing $h(n)$.

Hence $g(n)\geqq h(n)$, and so

$$g(n) = h(n).$$

THEOREM 2. *$g(n)=2^{n-1}f(n-1)$, for each n.*

Consider $G_n=(a_{rs})$ $(a_{rs}=\pm 1)$. If $a_{1s}\neq 1$, $a_{1s}=-1$ and in this case, by multiplying each element in the sth column by -1 we obtain

$$g(n) = \pm \begin{vmatrix} 1 & 1 & \cdots & 1 \\ & & b_{rs} & \end{vmatrix} \qquad b_{rs} = \pm 1, \qquad r \geqq 2.$$

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Similarly we can do the same for each element in the first column and obtain

$$g(n) = \pm \begin{vmatrix} 1 & 1 \cdots 1 \\ 1 & \\ 1 & \\ \vdots & c_{rs} \\ 1 & \end{vmatrix} \qquad c_{rs} = \pm 1, \quad r, s \geqq 2.$$

Therefore, interchanging the second and third rows if the sign outside is minus we obtain

$$g(n) = \begin{vmatrix} 1 & 1 \cdots 1 \\ 1 & \\ \vdots & d_{rs} \\ 1 & \end{vmatrix} \qquad d_{rs} = \pm 1, \quad r, s \geqq 2.$$

Now subtract the first row from each of the others and we obtain

$$g(n) = \begin{vmatrix} 1 & 1 \cdots 1 \\ 0 & \\ 0 & d_{rs} - 1 \\ \vdots & \\ 0 & \end{vmatrix}.$$

But if $d_{rs} = -1$ or 1, then $d_{rs} - 1 = 0$ or -2. Hence, expanding by the first column we obtain

$$g(n) = (-2)^{n-1} \left| e_{rs} \right|_{(n-1)} \quad \text{where} \quad e_{rs} = 0 \text{ or } 1.$$

Hence $g(n) \leqq 2^{n-1} f(n-1)$.

Conversely,

$$\begin{aligned} 2^{n-1} f(n-1) &= 2^{n-1} \left| a_{rs} \right|_{n-1} \qquad a_{rs} = 0 \text{ or } 1 \\ &= \left| 2a_{rs} \right|_{n-1} \end{aligned}$$

$$= \begin{vmatrix} & & 0 \\ & & 0 \\ & 2a_{rs} & 0 \\ & & \vdots \\ & & 0 \\ -1 & -1 \cdots -1 & 1 \end{vmatrix}_n = \begin{vmatrix} & & 1 \\ & & 1 \\ & 2a_{rs} - 1 & 1 \\ & & 1 \\ -1 & \cdots -1 & 1 \end{vmatrix}_n,$$

adding the last row to each of the others. But if $a_{rs}=0$ or 1, then

$$2a_{rs} - 1 = 1 \text{ or } -1 \quad \text{hence} \quad 2^{n-1}f(n-1) \leqq g(n).$$

This concludes the proof, and shows incidentally that $g(n)$ is always a multiple of 2^{n-1}.

This shows that the three problems are equivalent and so we shall concentrate on the second from now on.

We now prove the following results.

THEOREM 3. $g(n) \geqq (n-2)2^{n-1}$.

For $g(n)$ is not less than the circulant

$$\begin{vmatrix} 1 & 1 \cdots & 1 & -1 \\ -1 & 1 \cdots & 1 & 1 \\ 1 & -1 \cdots & 1 & 1 \\ \vdots & & & \\ 1 \cdots & \cdots & -1 & 1 \end{vmatrix} = (n-2)2^{n-1}.$$

THEOREM 4. $g(n+1) \geqq 2g(n)$.

For clearly $f(n) \geqq f(n-1)$ and so by Theorem 2, the result follows.

THEOREM 5. $g(n) \leqq n^{n/2}$.

This is an immediate corollary of Hadamard's inequality.

THEOREM 6. $g(1)=1$; $g(2)=2$.

The first of these is trivial. For the second we observe that

$$g(2) \geqq \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

and $g(2) \leqq 2$, by Theorem 5.

THEOREM 7. *For each* n, $g(2n) \geqq 2^n[g(n)]^2$.

Consider the $2n \times 2n$ matrix

$$\begin{pmatrix} G_n & -G_n \\ G_n & G_n \end{pmatrix}.$$

In this each element is 1 or -1.

Hence

$$g(2n) \geqq \begin{vmatrix} G_n & -G_n \\ G_n & G_n \end{vmatrix} = \begin{vmatrix} 2G_n & 0 \\ G_n & G_n \end{vmatrix}, = |2G_n| \cdot |G_n| = 2^n[g(n)]^2.$$

THEOREM 8. *If* $n=2^m$, $g(n)=n^{n/2}$.

By Theorem 5, $g(n)\leqq n^{n/2}$, and we prove by induction that $g(n)\geqq n^{n/2}$.

(a) We know that $g(2)=2$.

(b) Suppose that result is true for $n=2^{m_0}$.

Then, by Theorem 7

$$\begin{aligned} g(2n) &\geqq 2^n[g(n)]^2 \\ &\geqq 2^n[n^{n/2}]^2 \\ &= (2n)^n. \end{aligned}$$

This concludes the proof.

THEOREM 9. $g(mn)\geqq[g(m)]^n[g(n)]^m$.

Consider $g(n)=|G_n|=|a_{rs}|$, $a_{rs}=\pm 1$. Then it is well-known that by adding and subtracting rows and columns we may reduce this determinant to the diagonal form

$$g(n)=\begin{vmatrix} d_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & d_2 & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & d_n \end{vmatrix} = d_1d_2\cdots d_n.$$

Now consider the nmth order matrix

$$X=\begin{pmatrix} a_{11}I_m & a_{12}I_m & \cdots & a_{1n}I_m \\ \vdots & & & \\ a_{n1}I_m & \cdots & \cdots & a_{nn}I_m \end{pmatrix}.$$

Now the same process of adding rows and columns which diagonalised $g(n)$ will ensure that

$$|X| = \begin{vmatrix} d_1I_m & 0 & \cdots & 0 \\ 0 & d_2I_m & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & \cdots & \cdots & d_nI_m \end{vmatrix}$$

$$= d_1^m d_2^m \cdots d_n^m = [g(n)]^m.$$

Now consider

$$\begin{pmatrix} a_{11}I_m & a_{12}I_m & \cdots & a_{1n}I_m \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1}I_m & \cdots & \cdots & a_{nn}I_m \end{pmatrix} \begin{pmatrix} G_m & 0 & 0 \cdots 0 \\ 0 & G_m & \cdot \\ \cdot & & \cdot \\ \cdot & & 0 \\ 0 \cdots 0 & 0 & G_m \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}G_m & a_{12}G_m & \cdots & a_{1n}G_m \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1}G_m & \cdots & \cdots & a_{nn}G_m \end{pmatrix}.$$

Now since each $a_{rs} = \pm 1$, all the elements in the matrix on the R.H.S. are ± 1. Hence

$$g(nm) \geqq |X| \begin{vmatrix} G_m & 0 & \cdots & 0 \\ 0 & G_m & & \\ \cdot & & & \\ \cdot & & & G_m \end{vmatrix} = [g(n)]^m [g(m)]^n.$$

This proves the theorem.

THEOREM 10. $g(m^n) \geqq [g(m)]^{nm^{n-1}}$.

For by Theorem 9

$$\log g(m_1 m_2) \geqq m_1 \log g(m_2) + m_2 \log g(m_1).$$

Hence $\log g(m^2) \geqq 2m \log g(m)$. Suppose

$$\log g(m^k) \geqq km^{k-1} \log g(m).$$

Then

$$\begin{aligned} \log g(m^{k+1}) &\geqq mkm^{k-1} \log g(m) + m^k \log g(m) \\ &= (k+1)m^k \log g(m). \end{aligned}$$

Hence we have, by induction

$$g(m^n) \geqq [g(m)]^{nm^{n-1}}.$$

THEOREM 11. $g(m) \leqq mg(m-1)$.

For,

$$\begin{aligned} g(m) &= \begin{vmatrix} 1 & 1 \cdots 1 \\ & -1 \text{ or } 1 \end{vmatrix}_m \\ &= |\pm 1|_{m-1} - |\pm 1|_{m-1} \cdots (-)^{m-1} |\pm 1|_{m-1} \\ &\leqq mg(m-1). \end{aligned}$$

As an immediate corollary we have, by Theorem 8:

THEOREM 12. *If* $X=2^m$, $g(X-1)\geqq X^{X/2-1}$.

Our central theorem which we shall prove is

THEOREM 13. *For all sufficiently large* n, $g(n)\geqq n^{(1/2-\epsilon)n}$ *for any given positive* ϵ.

In order to prove this, we shall require the following lemmas.

LEMMA I. *For* $x>2$, $\xi(x)$ *is a monotonically increasing function, where*

$$\xi(x) = \frac{\log(x-1)}{\log\left(1+\dfrac{1}{x-1}\right)}$$

and $\xi(x)\to\infty$ *as* $x\to\infty$.

This is fairly obvious.

LEMMA II. *If* $n\geqq(x-1)^{\xi(x)+1}$ *then there exists an integer* α, *such that* $x^\alpha\geqq n\geqq(x-1)^\alpha$.

For, there certainly exists an integer α such that

$$(x-1)^{\alpha+1}\geqq n\geqq(x-1)^\alpha$$

and, moreover, $\alpha\geqq\xi(x)$ by the hypothesis. Hence

$$\frac{1}{\alpha}\leqq\frac{1}{\xi(x)}=\frac{\log\left(\dfrac{x}{x-1}\right)}{\log(x-1)}$$
$$=\frac{\log x-\log(x-1)}{\log(x-1)}$$

therefore, $(1+\alpha)\log(x-1)\leqq\alpha\log x$ and so $(x-1)^{\alpha+1}\leqq x^\alpha$. Hence $x^\alpha\geqq(x-1)^{\alpha+1}\geqq n\geqq(x-1)^\alpha$.

LEMMA III. *If* $\eta(x)=(a+x)/(b+cx)$ *where* b, c *are positive and* $1\leqq x\leqq d$ *then* $\eta(x)$ *reaches its lowest value, either when* $x=1$ *or when* $x=d$.

For $\eta'(x)$ has constant sign and is continuous for $1\leqq x\leqq d$.

LEMMA IV. *Given* n, *choose* $x=X$ *a power of* 2, *satisfying Lemma* II. *Then there exist integers* α, β *such that* $\alpha\geqq\xi(X)$ *and* $\alpha\geqq\beta\geqq1$ *and such that* $X^\beta(X-1)^{\alpha-\beta}\geqq n\geqq X^{\beta-1}(X-1)^{\alpha-\beta+1}$.

For by Lemma II, $\alpha \geqq \xi(X)$ and $X^\alpha \geqq n \geqq (X-1)^\alpha$. Hence

$$\left(\frac{X}{X-1}\right)^\alpha \geqq \frac{n}{(X-1)^\alpha} \geqq 1.$$

Hence there exists an integer β, such that $(X/(X-1))^\beta \geqq n/(X-1)^\alpha \geqq (X/(X-1))^{\beta-1}$ and $\alpha \geqq \beta \geqq 1$. Hence $X^\beta(X-1)^{\alpha-\beta} \geqq n \geqq (X-1)^{\alpha-\beta+1}X^{\beta-1}$.

We are now in a position to complete the proof. We have by Lemma IV,

$$g(n) \geqq g[(X-1)^{\alpha-\beta+1}X^{\beta-1}].$$

Hence by Theorem 9

$$\begin{aligned}\log g(n) &\geqq (X-1)^{\alpha-\beta+1}\log g(X^{\beta-1}) + X^{\beta-1}\log g\{(X-1)^{\alpha-\beta+1}\}\\ &\geqq (X-1)^{\alpha-\beta+1}(\beta-1)X^{\beta-2}\log g(X)\\ &\quad + X^{\beta-1}(\alpha-\beta+1)(X-1)^{\alpha-\beta}\log (X-1), \text{ by Theorem 10,}\\ &\geqq (X-1)^{\alpha-\beta+1}(\beta-1)X^{\beta-2}\tfrac{1}{2}X\log X\\ &\quad + X^{\beta-1}(\alpha-\beta+1)(X-1)^{\alpha-\beta}(\tfrac{1}{2}X-1)\log X\end{aligned}$$

by Theorems 8 and 12, since X is a power of 2. Hence

$$\begin{aligned}&\log g(n)\\ &\quad \geqq \tfrac{1}{2}\log X\cdot(X-1)^{\alpha-\beta}X^{\beta-1}[(\alpha-\beta+1)(X-2)+(\beta-1)(X-1)]\\ &\quad \log g(n) \geqq \tfrac{1}{2}\log X\cdot(X-1)^{\alpha-\beta}X^{\beta-1}[\alpha(X-2)-1+\beta].\end{aligned}$$

Also $X^\beta(X-1)^{\alpha-\beta} \geqq n$ and so

$$n\log n \leqq X^\beta(X-1)^{\alpha-\beta}[(\alpha-\beta)\log(X-1)+\beta\log X].$$

Hence

$$\frac{\log g(n)}{n\log n} \geqq \frac{\log X}{2X}\,\frac{\alpha(X-2)-1+\beta}{\alpha\log(X-1)+\beta\log\left(\dfrac{X}{X-1}\right)}.$$

Now $\alpha \geqq \beta \geqq 1$ and so by Lemma III the lowest value of the expression on the right hand side occurs when *either* $\beta = 1$ *or* $\beta = \alpha$. Hence

$$\frac{\log g(n)}{n\log n} \geqq \min\{A, B\},$$

where

$$A = \frac{\log X}{2X} \frac{\alpha(X-2)}{\alpha \log(X-1) + \log\left(\frac{X}{X-1}\right)},$$

$$B = \frac{\log X}{2X} \frac{\alpha(X-1)-1}{\alpha \log X} = \frac{\alpha(X-1)-1}{2X\alpha}.$$

Now $B = \frac{1}{2} - (1+\alpha)/2X\alpha$, but $\alpha \geqq \xi(X) > 1$ hence $1 + 1/\alpha < 2$, and $B > \frac{1}{2} - 1/X > \frac{1}{2} - \epsilon$ provided $X > 1/\epsilon$.

But, given $\epsilon > 0$ we may choose a power X of 2 such that $X > 1/\epsilon$. Then for every $n \geqq n_0 = (X-1)^{\xi(X)+1}$ we have the above inequality. Also

$$A = \frac{\log X}{2X} \cdot \frac{X-2}{\log(X-1)} \cdot \left\{1 - \frac{\log \frac{X}{X-1}}{\alpha \log(X-1) + \log\left(\frac{X}{X-1}\right)}\right\}$$

$$> \frac{\log X}{2X} \cdot \frac{X-2}{\log(X-1)} \left\{1 - \frac{\log \frac{X}{X-1}}{\log(X-1) + \log \frac{X}{X-1}}\right\}$$

$$= \frac{\log X}{2X} \cdot \frac{X-2}{\log(X-1)} \left\{\frac{\log(X-1)}{\log X}\right\}.$$

Hence $A > \frac{1}{2} - 1/X > \frac{1}{2} - \epsilon$, and for all sufficiently large n,

$$\frac{\log g(n)}{n \log n} > \frac{1}{2} - \epsilon$$

i.e., $g(n) > n^{n(1/2-\epsilon)}$ which concludes the proof of Theorem 13.

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