THE RAYLEIGH FUNCTION

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The present paper is a study of a set of symmetric functions of the zeros of $J_\nu(z)$, the Bessel function of the first kind. Let the zeros of $z^{-\nu}J_\nu(z)$ be denoted by $j_{\nu,m}$, $m=1, 2, \cdots$, where $|R(j_{\nu,m})| \leqq |R(j_{\nu,m+1})|$. And let

$$\sigma_{2n}(\nu) \equiv \sum_{m=1}^{\infty} (j_{\nu,m})^{-2n}, \qquad n = 1, 2, \cdots. \tag{1}$$

We propose to call $\sigma_{2n}(\nu)$ the Rayleigh function of order $2n$. The Rayleigh functions of odd orders are identically zero.

These functions were first used by Euler to determine the three smallest zeros of $J_0(2\sqrt{z})$; and Rayleigh, independently, calculated the smallest positive zero of $J_\nu(z)$ with the aid of these functions. The Rayleigh functions have been the subject of a number of investigations by Cayley, Graeffe, Graf and Gubler, Watson, Kapteyn, Forsyth and others [6, p. 502].

In this paper we shall understand by the Bernoulli and Genocchi numbers, the entities B_n and G_n defined as follows:

$$\begin{aligned} &B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \; n \neq 1, \\ &B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_3 = 0, \quad B_4 = -\tfrac{1}{30}, \cdots \end{aligned} \tag{2}$$

$$G_n = 2(1 - 2^n)B_n; \tag{3}$$

(see [4; 3]).

From the well-known formula

$$z^{-1/2}J_{1/2}(z) = \sqrt{\frac{2}{\pi}} \sin z/z,$$

(see [6, p. 54]), the roots of $z^{-1/2}J_{1/2}(z)$ are seen to be $k\pi$, $k=1, 2, \cdots$. Hence

$$\sigma_{2n}\left(\frac{1}{2}\right) = \sum_{k=1}^{\infty} (k\pi)^{-2n} = \pi^{-2n}\zeta(2n) = (-1)^{n-1}\frac{2^{2n-1}}{(2n)!} B_{2n}. \tag{4}$$

Similarly, since $z^{1/2}J_{-1/2}(z) = \sqrt{2/\pi} \cos z$, the roots of $z^{1/2}J_{-1/2}(z)$ are

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$(k-\frac{1}{2})\pi$, $k=1, 2, \cdots$. Therefore,

(5)
$$\sigma_{2n}\left(-\frac{1}{2}\right)=\pi^{-2n}\sum_{k=1}^{\infty}\left(k-\frac{1}{2}\right)^{-2n}=(2^{2n}-1)\pi^{-2n}\zeta(2n)$$
$$=(-1)^n\frac{2^{2n-2}}{(2n)!}G_{2n}.$$

A generating function for $\sigma_{2n}(\nu)$ can be obtained. Let $J_\nu(z)$ be expressed as a Weierstrassian product

$$J_\nu(z)=\frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\prod_{n=1}^{\infty}\left\{1-\frac{z^2}{j_{\nu,n}^2}\right\},$$

(see [6, p. 498]). Differentiating logarithmically with respect to z,

$$J_\nu'(z)/J_\nu(z)=\nu/z+\sum_{k=1}^{\infty}(-2z)/(j_{\nu,k}^2-z^2)$$
$$=1/z\left\{\nu-2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}z^{2n}/j_{\nu,k}^{2n}\right\},$$

$$zJ_\nu'(z)/J_\nu(z)=\nu-2\sum_{n=1}^{\infty}\sigma_{2n}(\nu)z^{2n},$$

(6)
$$\frac{1}{2}zJ_{\nu+1}(z)/J_\nu(z)=\sum_{n=1}^{\infty}\sigma_{2n}(\nu)z^{2n}.$$

Substituting $z=1$ in (6) yields

(7)
$$\sum_{n=1}^{\infty}\sigma_{2n}(\nu)=\frac{1}{2}J_{\nu+1}(1)/J_\nu(1);$$

(8)
$$\sum_{n=1}^{\infty}\sigma_{2n}(\nu+k)=\frac{1}{2}J_{\nu+k+1}(1)/J_{\nu+k}(1),\qquad k=0,1,2,\cdots.$$

Setting $\nu=+\frac{1}{2}$ and $-\frac{1}{2}$ in (6), the following are obtained in view of (4) and (5),

(9)
$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}}{(2n)!}B_{2n}z^{2n}=1-z\cot z,$$

(10)
$$\sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}}{(2n)!}G_{2n}z^{2n}=z\tan z.$$

Taking the continued fraction representation of the ratio of two

Bessel functions,

$$J_{\nu+1}(z)/J_\nu(z) = \frac{z}{2(\nu+1)-}\ \frac{z^2}{2(\nu+2)-}\cdots,$$

(see [6, p. 153)] then substituting $z=1$, and using (7),

$$\text{(11)} \qquad \sum_{n=1}^{\infty} \sigma_{2n}(\nu) = \frac{1}{2}\cdot\frac{1}{2(\nu+1)-}\ \frac{1}{2(\nu+2)-}\cdots.$$

Set $\nu=\pm\frac{1}{2}$ in (11), then

$$\text{(12)} \qquad \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2^{2n}}{(2n)!}B_{2n} = \frac{1}{3-}\ \frac{1}{5-}\ \frac{1}{7-}\cdots$$

$$= 1-\cot(1) = .3579,$$

$$\text{(13)} \qquad \sum_{n=1}^{\infty} (-1)^{n}\frac{2^{2n-1}}{(2n)!}G_{2n} = \frac{1}{1-}\ \frac{1}{3-}\ \frac{1}{5-}\cdots$$

$$= \tan(1) = 1.5574.$$

A recurrence formula for the functions $\sigma_{2n}(\nu)$ may be derived from (6). Let (6), be written as

$$\frac{1}{2}zJ_{\nu+1}(z) = J_\nu(z)\sum_{n=1}^{\infty}\sigma_{2n}(\nu)z^{2n}.$$

Substituting the series for $J_\nu(z)$ and $J_{\nu+1}(z)$, and identifying the coefficients of z^{2n} on both sides,

$$\text{(14)} \qquad \sum_{k=1}^{n}(-1)^{k-1}4^k(k!)^2\binom{n}{k}\binom{\nu+n}{k}\sigma_{2k}(\nu) = n.$$

Substitute $\nu=\pm\frac{1}{2}$ in (14), then in view of (4) and (5),

$$\text{(15)} \qquad \sum_{k=1}^{n}2^{2k-1}\binom{2n+1}{k}B_{2k} = n,$$

(see [5, p. 174])

$$\text{(16)} \qquad -\sum_{k=1}^{n}2^{2k-2}\binom{2n}{2k}G_{2k} = n;$$

which are well-known Bernoulli and Genocchi recurrence formulas.

A determinant representation of $\sigma_{2n}(\nu)$ may be given. If in (14) the upper limit of the summation is taken as $1, 2, \cdots, n$, then n linear equations involving n functions $\sigma_2(\nu), \sigma_4(\nu), \cdots, \sigma_{2n}(\nu)$ are obtained. The determinant Δ of this system is triangular. Hence, the value of Δ is equal to the product of its elements on the main diagonal,

$$\Delta = (-1)^{n(n-1)/2} \cdot 2^{n(n+1)} \cdot n!! \prod_{k=1}^{n} (\nu + k)^{n-k+1}.$$

Then by Cramer's rule,

$$\sigma_{2n}(\nu) = (-1)^{n-1} \frac{(n-1)!!}{n!} 4^{-n} \prod_{k=1}^{n} (\nu + k)^{-n+k-1} \cdot D, \tag{17}$$

where,

$$D = \begin{vmatrix} \binom{1}{1}\binom{\nu+1}{1} & 0 & 0 & \cdots & 0 & 1 \\ \binom{2}{1}\binom{\nu+2}{1} & \binom{2}{2}\binom{\nu+2}{2} & 0 & \cdots & 0 & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \binom{n}{1}\binom{\nu+n}{1} & \binom{n}{2}\binom{\nu+n}{2} & \cdots & & \binom{n}{n-1}\binom{\nu+n}{n-1} & n \end{vmatrix}.$$

Substitution of $\nu = \pm\frac{1}{2}$ in (17) yields, respectively,

$$B_{2n} = \frac{n!}{(2n+1)!} 2^{-n+1} D_1, \tag{18}$$

$$G_{2n} = -\, 2^{-2n+2} D_2, \tag{19}$$

where

$$D_1 = \begin{vmatrix} \binom{3}{2} & 0 & 0 \cdots 0 & 1 \\ \binom{5}{2} & \binom{5}{4} & 0 \cdots 0 & 2 \\ \cdot & \cdot & \cdot & \cdot \\ \binom{2n+1}{2} & \cdots & \binom{2n+1}{2n-2} & n \end{vmatrix},$$

$$D_2 = \begin{vmatrix} \binom{2}{2} & 0 & 0 \cdots 0 & 1 \\ \binom{4}{2} & \binom{4}{4} & 0 \cdots 0 & 2 \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \\ \binom{2n}{2} & \cdots & \binom{2n}{2n-2} & n \end{vmatrix}.$$

Other formulas for $\sigma_{2n}(\nu)$ may be derived. Let (6) be written as

$$\{zJ_{\nu+2}(z)/J_{\nu+1}(z)\}\{zJ_{\nu+1}(z)/J_\nu(z)\} = 4\left\{\sum_{k=1}^{\infty}\sigma_{2k}(\nu+1)z^{2k}\right\}\left\{\sum_{k=1}^{\infty}\sigma_{2k}(\nu)z^{2k}\right\}.$$

Then in view of the well-known formula,

$$J_{\nu+2}(z) = \frac{2(\nu+1)}{z}J_{\nu+1}(z) - J_\nu(z),$$

(see [6, p. 45]), the above becomes

$$-\frac{1}{4}z^2 + \frac{1}{2}(\nu+1)zJ_{\nu+1}(z)/J_\nu(z) = \left\{\sum_{k=1}^{\infty}\sigma_{2k}(\nu+1)z^{2k}\right\}\left\{\sum_{k=1}^{\infty}\sigma_{2k}(\nu)z^{2k}\right\}.$$

Identifying the coefficients of z^{2n} on both sides

$$(\nu+1)\sigma_{2n}(\nu) = \sum_{k=1}^{n-1}\sigma_{2k}(\nu+1)\sigma_{2n-2k}(\nu). \tag{20}$$

Substitution of $\nu = -\frac{1}{2}$, in (20) yields

$$G_{2n} = -\sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}G_{2n-2k}. \tag{21}$$

Consider the well-known formulas

$$\frac{d}{dz}(z^{-\nu}J_\nu(z)) = -z^{-\nu}J_{\nu+1}(z),$$

(see [6, p. 45]),

$$\frac{d}{dz}(z^{\nu+1}J_{\nu+1}(z)) = z^{\nu+1}J_\nu(z).$$

It follows that

$$\frac{d}{dz}\left(\frac{z^{\nu+1}J_{\nu+1}(z)}{z^{-\nu}J_\nu(z)}\right) = z^{2\nu+1}\frac{(1 + J_{\nu+1}^2(z)}{J_\nu^2(z)}$$

That is,

$$\left(\frac{z}{2}\frac{J_{\nu+1}(z)}{J_\nu(z)}\right)^2 = -\frac{z^2}{4} + \frac{1}{2}z^{-2\nu+1}\frac{d}{dz}\left(\frac{z}{2}\frac{J_{\nu+1}(z)}{J_\nu(z)}z^{2\nu}\right).$$

Substituting (6) in this relation and identifying the coefficients of z^{2n},

$$(\nu + n)\sigma_{2n}(\nu) = \sum_{k=1}^{n-1}\sigma_{2k}(\nu)\sigma_{2n-2k}(\nu). \tag{22}$$

Substitute $\nu = \pm\frac{1}{2}$ in (22), then

$$-(1 + 2n)B_{2n} = \sum_{k=1}^{n-1}\binom{2n}{2k}B_{2k}B_{2n-2k}, \tag{23}$$

(see [5, p. 66])

$$-2(1 - 2n)G_{2n} = \sum_{k=1}^{n-1}\binom{2n}{2k}G_{2k}G_{2n-2k}. \tag{24}$$

Let (6) be multiplied by itself,

$$J_{\nu+1}^2(z) = 4J_\nu^2(z)\left(\sum_{k=1}^{\infty}\sigma_{2k}(\nu)z^{2k-1}\right)^2,$$

then substituting the well-known series for $J_{\nu+1}^2(z)$ and $J_\nu^2(z)$, viz.,

$$J_{\nu+1}^2(z) = \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(2\nu + 3 + 2k)}{k!\Gamma(2\nu + 3 + k)\Gamma^2(\nu + 2 + k)}\left(\frac{z}{2}\right)^{2\nu+2+2k},$$

$$J_\nu^2(z) = \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(2\nu + 1 + 2k)}{k!\Gamma(2\nu + 1 + k)\Gamma^2(\nu + 1 + k)}\left(\frac{z}{2}\right)^{2\nu+2k},$$

(see [6, p. 147]), and identifying the coefficients of z^{2n} on both sides, the following is obtained

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k-1}\binom{2\nu + 2n - 2k}{n - k}(k!)^2\binom{\nu + n}{k}^2 4^{k+1}\sum_{s=1}^{k}\sigma_{2s}(\nu)\sigma_{2(k-s+1)}(\nu)\\ = \binom{2\nu + 2n}{n - 1}.\end{aligned} \tag{25}$$

In view of (22) this reduces to

$$\sum_{k=1}^{n}(-1)^{k-1}4^k\binom{2\nu+2n-2k}{n-k}\{(k-1)!\}^2\binom{\nu+n-1}{k-1}^2(\nu+R)\,\sigma_{2k}(\nu) \tag{26}$$
$$=\frac{2\nu+1}{2\nu+n}\binom{2\nu+2n-2}{n-1}.$$

Substitute $\nu=\pm\frac{1}{2}$ in (26), then

$$\sum_{k=1}^{n-1}(1+2k)\binom{2n}{2k}B_{2k}=2n-1, \tag{27}$$

$$\sum_{k=1}^{n-1}(1-2k)\binom{2n}{2k}G_{2k}=2(2n-1)G_{2n}. \tag{28}$$

The Rayleigh functions of odd order are zero,

$$\sigma_1(\nu)=0,\quad \sigma_3(\nu)=0,\quad \sigma_5(\nu)=0,\cdots.$$

Identifying the coefficients of z^2 in (6), an explicit expression for $\sigma_2(\nu)$ is obtained. Then (20) may be used to get explicit expressions for the Rayleigh functions of higher orders. Thus,

$$\sigma_2(\nu)=\frac{1}{2^2(\nu+1)};$$

$$\sigma_4(\nu)=\frac{1}{2^4(\nu+1)^2(\nu+2)};$$

$$\sigma_6(\nu)=\frac{2}{2^6(\nu+1)^3(\nu+2)(\nu+3)};$$

$$\sigma_8(\nu)=\frac{5\nu+11}{2^8(\nu+1)^4(\nu+2)^2(\nu+3)(\nu+4)}.$$

The first twelve Rayleigh functions are given in [1, pp. 405–407].

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