MAZUR'S THEOREM ON SEQUENTIALLY CONTINUOUS LINEAR FUNCTIONALS

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1. Introduction. A linear topological space A *satisfies Mazur's theorem* provided every sequentially continuous linear functional on A is continuous. There have been a number of investigations of conditions on a space A of real valued functions on a set X in order that A, with the topology of pointwise convergence on X, should satisfy Mazur's theorem. S. Mazur opened the question for spaces $A = C(X)$ and gave a strong theorem [**3**] whose statement is a little too complicated to give here. The fact that Mazur's theorem always holds for $C(X)$ if X is compact seems first to have been proved by V. Pták in 1956 (see [**2**]); the first publication seems to be in [**4**, Theorem C and footnote 8].

A linear space of functions A on a set X determines a uniform structure on X. Relative to this structure X has a completion, $\hat{X}$. As is well known, a linear functional on A which is continuous relative to the weak topology is *representable in* X, i.e., a finite linear combination of evaluations at points of X.

In this paper we split the problem of Mazur's theorem into two parts as follows. (1) For which spaces A is every sequentially continuous linear functional on A representable in $\hat{X}$? (2) Which points p of $\hat{X}$ yield sequentially continuous linear functionals? We give three theorems, all assuming that A is closed under certain operations. The weakest assumptions are, alternatively, that A is closed under the lattice operations $f \vee g$, $f \wedge g$, and the bounding operations $(f \wedge n) \vee -n$, or that A is closed under composition with real entire functions vanishing at 0; either assumption takes care of problem (1). For (2) we assume also that A contains the constant functions; then the points p in question are those for which every countable set of functions in A vanishing at p has a common zero in X. Finally, replacing the entire functions with C^∞ functions, we show that if every function locally belonging to A belongs to A then evaluation at any point of $\hat{X}$ is sequentially continuous.

We remark that examples in [**2**] show that some linear subspaces of $C(X)$ may admit sequentially continuous linear functionals which are not representable in $\hat{X}$; also, that the two papers by Isbell and Thomas, respectively, which are cited in [**2**], are combined in this paper.

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2. **Theorems.** We say A is *lattice-closed* if A is closed under the two unary operations $|f|$, $(f\wedge 1)\vee -1$; since we are also assuming A is a linear space, it is closed under $f\vee g$, $f\wedge g$, and $(f\wedge n)\vee -n$ as well. We say A is *analytically closed* if it is closed under all the unary operations $h\circ f$, where h is an entire function real on the real line and 0 at 0. A is *locally determined* if every function defined on X which agrees with some function in A on a neighborhood of each point belongs to A. We assume for convenience that A separates points on X.

The entire functions to be used below are as follows: Functions α_n within $1/n$ of $|x|$ on $[-n, n]$ (these may be polynomials); functions β_n within $1/n$ of $(x\wedge n)\vee -n$ on the whole line; the square function $g(x)=x^2$; a function h with $h(0)=h(+\infty)=0\neq h(-1)$. The existence of the β_n follows from Carleman's approximation theorem [1].

We use a special case of a lemma of H. H. Corson (see [2]); if A and B are linear spaces of functions on a set X, and A is a uniformly dense subset of B, then every sequentially continuous linear functional on A in the weak topology has a sequentially continuous extension over B.

THEOREM 1. *If A is* (i) *lattice-closed or* (ii) *analytically closed, then every sequentially continuous linear functional on A is representable in $\hat{X}$.*

PROOF. Let A^* be the subspace of A consisting of all the bounded functions. A^* determines a uniform structure on X and a compact completion $\overline{X}$; so we may regard A^* as contained in $C(\overline{X})$. Either hypothesis implies (using the functions α_n) that the closure of A^* in the norm topology is a lattice. Therefore it is either $C(\overline{X})$ or a closed hyperplane consisting of all functions vanishing at a point.

Then for any sequentially continuous functional ϕ on A, the restriction $\phi|A^*$ has at least one sequentially continuous extension over $C(\overline{X})$ in the topology of pointwise convergence on X. Since the topology of pointwise convergence on $\overline{X}$ is finer, it follows from Pták's theorem [2;4] that $\phi|A^*$ is representable in $\overline{X}$. To conclude, it suffices to show that if ψ is a sequentially continuous linear functional on A and p is a point of $\overline{X}$ such that $\psi(f)=f(p)$ for all f in A^*, then there is a net of points x_λ of X such that $f(x_\lambda)$ converges to $\psi(f)$ for all f in A. (That is, the "good" points of $\overline{X}$ are points of $\hat{X}$.) For this we take any net $\{x_\lambda\}$ converging to p in $\overline{X}$; and we consider the functions f_n defined (case (i)) as $(f\wedge n)\vee -n$, or (case (ii)) as $\beta_n\circ f$. For any positive ϵ, choose an index $m>\epsilon^{-1}$ such that $m-1/m>|\psi(f)|+\epsilon$, and $|\psi(f_m)-\psi(f)|<\epsilon$. The numbers $f_m(x_\lambda)$ converge to $\psi(f_m)$;

so they are finally within ϵ of $\psi(f)$. In particular $|f_m(x_\lambda)|<m-1/m$, finally in λ. Hence for these λ, $|f_m(x_\lambda)-f(x_\lambda)|<1/m$. Thus $|f(x_\lambda)-\psi(f)|<\epsilon+1/m<2\epsilon$. Since ϵ is arbitrary, this proves $f(x_\lambda)\to\psi(f)$, as was to be shown.

THEOREM 2. *Let A be as in Theorem* 1 *and contain the constants, and let p be a point of $\hat{X}$. Then evaluation at p is sequentially continuous on A if and only if every sequence in A vanishing identically at p vanishes identically at some point of X.*

PROOF. Assume (ii) that A is analytically closed, and suppose $\{f_n\}$ is a sequence vanishing at p but, for each x in X, some $f_n(x)\neq 0$. Then defining g_n as $n\sum_{i=1}^{i=n} f_i^2$, we have $g_n(p)=0$ but $g_n\to+\infty$ on X. If A contains the constants, define $h_n=h\circ(g_n-1)$, where $h(0)=h(+\infty)=0$, $h(-1)\neq 0$. Then $h_n\to 0$ on X but $h_n(p)=h(-1)$; so evaluation at p is not sequentially continuous.

Conversely, if evaluation at p is not sequentially continuous, we may pick a sequence of non-negative functions f_n converging to 0 on X but with $f_n(p)=1$ for all n. Setting $j_n(x)=1-f_n(x)$, we have a sequence all vanishing at p but not all vanishing at any point of X

For case (i), we simply use $|x|$ instead of x^2 and $x\wedge 0$ instead of h.

THEOREM 3. *If A is locally determined and* (i) *lattice-closed or* (ii) *closed under composition with real C^∞ functions vanishing at* 0, *then evaluation at each point of $\hat{X}$ is sequentially continuous.*

PROOF. We show first that the constants can be adjoined to A, and the hypotheses preserved, without changing $\hat{X}$; in fact we need only take the functions $f+k$, f in A, k constant. Since A separates points on X, there is at most one point x_0 at which all functions in A vanish. Suppose g is a function on X which locally has the form $f_\alpha+k_\alpha$, for various f_α in A and various constants k_α. If $g=f_0+k_0$ near x_0, then $g-k_0$ agrees with a function in A near x_0, and also everywhere else; for at any other point x of X there is a function in A taking a nonzero value and, by (i) or (ii), a function in A taking a nonzero constant value near x. Thus the space of all $f+k$ is locally determined. Entirely similar arguments show that it retains property (i) or (ii). So we may assume A already contains the constants.

Suppose for some p in $\hat{X}$ that evaluation at p is not sequentially continuous. As in the proof of Theorem 2, there is a sequence of non-negative functions f_n in A, each vanishing at p, with $f_n(x)$ forming an unbounded increasing sequence for each x in X. Let g be a non-negative, real, C^∞ function vanishing on $(-\infty, 1/2]$ and identically 1 on $[1, \infty)$, and consider any x in X. Near x almost all $f_n\geqq 1$, hence the equation $f(x)=\sum(1-g\circ f_n(x))$ defines a function which, near any x in X, is a finite sum of functions in A and therefore is in A.

On the other hand f cannot be extended over p, for near p arbitrarily many $g\circ f_n$ are zero and f is arbitrarily large. The contradiction establishes the theorem.

References

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