# MAZUR'S THEOREM ON SEQUENTIALLY CONTINUOUS LINEAR FUNCTIONALS

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**1. Introduction.** A linear topological space $A$ *satisfies Mazur's theorem* provided every sequentially continuous linear functional on $A$ is continuous. There have been a number of investigations of conditions on a space $A$ of real valued functions on a set $X$ in order that $A$, with the topology of pointwise convergence on $X$, should satisfy Mazur's theorem. S. Mazur opened the question for spaces $A = C(X)$ and gave a strong theorem [**3**] whose statement is a little too complicated to give here. The fact that Mazur's theorem always holds for $C(X)$ if $X$ is compact seems first to have been proved by V. Pták in 1956 (see [**2**]); the first publication seems to be in [**4**, Theorem C and footnote 8].

A linear space of functions $A$ on a set $X$ determines a uniform structure on $X$. Relative to this structure $X$ has a completion, $\hat{X}$. As is well known, a linear functional on $A$ which is continuous relative to the weak topology is *representable in* $X$, i.e., a finite linear combination of evaluations at points of $X$.

In this paper we split the problem of Mazur's theorem into two parts as follows. (1) For which spaces $A$ is every sequentially continuous linear functional on $A$ representable in $\hat{X}$? (2) Which points $p$ of $\hat{X}$ yield sequentially continuous linear functionals? We give three theorems, all assuming that $A$ is closed under certain operations. The weakest assumptions are, alternatively, that $A$ is closed under the lattice operations $f \vee g$, $f \wedge g$, and the bounding operations $(f \wedge n) \vee -n$, or that $A$ is closed under composition with real entire functions vanishing at 0; either assumption takes care of problem (1). For (2) we assume also that $A$ contains the constant functions; then the points $p$ in question are those for which every countable set of functions in $A$ vanishing at $p$ has a common zero in $X$. Finally, replacing the entire functions with $C^\infty$ functions, we show that if every function locally belonging to $A$ belongs to $A$ then evaluation at any point of $\hat{X}$ is sequentially continuous.

We remark that examples in [**2**] show that some linear subspaces of $C(X)$ may admit sequentially continuous linear functionals which are not representable in $\hat{X}$; also, that the two papers by Isbell and Thomas, respectively, which are cited in [**2**], are combined in this paper.

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2. **Theorems.** We say $A$ is *lattice-closed* if $A$ is closed under the two unary operations $|f|$, $(f\wedge 1)\vee -1$; since we are also assuming $A$ is a linear space, it is closed under $f\vee g$, $f\wedge g$, and $(f\wedge n)\vee -n$ as well. We say $A$ is *analytically closed* if it is closed under all the unary operations $h\circ f$, where $h$ is an entire function real on the real line and 0 at 0. $A$ is *locally determined* if every function defined on $X$ which agrees with some function in $A$ on a neighborhood of each point belongs to $A$. We assume for convenience that $A$ separates points on $X$.

The entire functions to be used below are as follows: Functions $\alpha_n$ within $1/n$ of $|x|$ on $[-n, n]$ (these may be polynomials); functions $\beta_n$ within $1/n$ of $(x\wedge n)\vee -n$ on the whole line; the square function $g(x)=x^2$; a function $h$ with $h(0)=h(+\infty)=0\neq h(-1)$. The existence of the $\beta_n$ follows from Carleman's approximation theorem [1].

We use a special case of a lemma of H. H. Corson (see [2]); if $A$ and $B$ are linear spaces of functions on a set $X$, and $A$ is a uniformly dense subset of $B$, then every sequentially continuous linear functional on $A$ in the weak topology has a sequentially continuous extension over $B$.

THEOREM 1. *If $A$ is* (i) *lattice-closed or* (ii) *analytically closed, then every sequentially continuous linear functional on $A$ is representable in $\hat{X}$.*

PROOF. Let $A^*$ be the subspace of $A$ consisting of all the bounded functions. $A^*$ determines a uniform structure on $X$ and a compact completion $\overline{X}$; so we may regard $A^*$ as contained in $C(\overline{X})$. Either hypothesis implies (using the functions $\alpha_n$) that the closure of $A^*$ in the norm topology is a lattice. Therefore it is either $C(\overline{X})$ or a closed hyperplane consisting of all functions vanishing at a point.

Then for any sequentially continuous functional $\phi$ on $A$, the restriction $\phi|A^*$ has at least one sequentially continuous extension over $C(\overline{X})$ in the topology of pointwise convergence on $X$. Since the topology of pointwise convergence on $\overline{X}$ is finer, it follows from Pták's theorem [2;4] that $\phi|A^*$ is representable in $\overline{X}$. To conclude, it suffices to show that if $\psi$ is a sequentially continuous linear functional on $A$ and $p$ is a point of $\overline{X}$ such that $\psi(f)=f(p)$ for all $f$ in $A^*$, then there is a net of points $x_\lambda$ of $X$ such that $f(x_\lambda)$ converges to $\psi(f)$ for all $f$ in $A$. (That is, the "good" points of $\overline{X}$ are points of $\hat{X}$.) For this we take any net $\{x_\lambda\}$ converging to $p$ in $\overline{X}$; and we consider the functions $f_n$ defined (case (i)) as $(f\wedge n)\vee -n$, or (case (ii)) as $\beta_n\circ f$. For any positive $\epsilon$, choose an index $m>\epsilon^{-1}$ such that $m-1/m>|\psi(f)|+\epsilon$, and $|\psi(f_m)-\psi(f)|<\epsilon$. The numbers $f_m(x_\lambda)$ converge to $\psi(f_m)$;

so they are finally within $\epsilon$ of $\psi(f)$. In particular $|f_m(x_\lambda)| < m - 1/m$, finally in $\lambda$. Hence for these $\lambda$, $|f_m(x_\lambda) - f(x_\lambda)| < 1/m$. Thus $|f(x_\lambda) - \psi(f)| < \epsilon + 1/m < 2\epsilon$. Since $\epsilon$ is arbitrary, this proves $f(x_\lambda) \to \psi(f)$, as was to be shown.

THEOREM 2. *Let $A$ be as in Theorem* 1 *and contain the constants, and let $p$ be a point of $\hat{X}$. Then evaluation at $p$ is sequentially continuous on $A$ if and only if every sequence in $A$ vanishing identically at $p$ vanishes identically at some point of $X$.*

PROOF. Assume (ii) that $A$ is analytically closed, and suppose $\{f_n\}$ is a sequence vanishing at $p$ but, for each $x$ in $X$, some $f_n(x) \neq 0$. Then defining $g_n$ as $n \sum_{i=1}^{i=n} f_i^2$, we have $g_n(p) = 0$ but $g_n \to +\infty$ on $X$. If $A$ contains the constants, define $h_n = h \circ (g_n - 1)$, where $h(0) = h(+\infty) = 0$, $h(-1) \neq 0$. Then $h_n \to 0$ on $X$ but $h_n(p) = h(-1)$; so evaluation at $p$ is not sequentially continuous.

Conversely, if evaluation at $p$ is not sequentially continuous, we may pick a sequence of non-negative functions $f_n$ converging to 0 on $X$ but with $f_n(p) = 1$ for all $n$. Setting $j_n(x) = 1 - f_n(x)$, we have a sequence all vanishing at $p$ but not all vanishing at any point of $X$.

For case (i), we simply use $|x|$ instead of $x^2$ and $x \wedge 0$ instead of $h$.

THEOREM 3. *If $A$ is locally determined and* (i) *lattice-closed or* (ii) *closed under composition with real $C^\infty$ functions vanishing at* 0, *then evaluation at each point of $\hat{X}$ is sequentially continuous.*

PROOF. We show first that the constants can be adjoined to $A$, and the hypotheses preserved, without changing $\hat{X}$; in fact we need only take the functions $f + k$, $f$ in $A$, $k$ constant. Since $A$ separates points on $X$, there is at most one point $x_0$ at which all functions in $A$ vanish. Suppose $g$ is a function on $X$ which locally has the form $f_\alpha + k_\alpha$, for various $f_\alpha$ in $A$ and various constants $k_\alpha$. If $g = f_0 + k_0$ near $x_0$, then $g - k_0$ agrees with a function in $A$ near $x_0$, and also everywhere else; for at any other point $x$ of $X$ there is a function in $A$ taking a nonzero value and, by (i) or (ii), a function in $A$ taking a nonzero constant value near $x$. Thus the space of all $f + k$ is locally determined. Entirely similar arguments show that it retains property (i) or (ii). So we may assume $A$ already contains the constants.

Suppose for some $p$ in $\hat{X}$ that evaluation at $p$ is not sequentially continuous. As in the proof of Theorem 2, there is a sequence of non-negative functions $f_n$ in $A$, each vanishing at $p$, with $f_n(x)$ forming an unbounded increasing sequence for each $x$ in $X$. Let $g$ be a non-negative, real, $C^\infty$ function vanishing on $(-\infty, 1/2]$ and identically 1 on $[1, \infty)$, and consider any $x$ in $X$. Near $x$ almost all $f_n \geqq 1$, hence the equation $f(x) = \sum(1 - g \circ f_n(x))$ defines a function which, near any $x$ in $X$, is a finite sum of functions in $A$ and therefore is in $A$.

On the other hand $f$ cannot be extended over $p$, for near $p$ arbitrarily many $g \circ f_n$ are zero and $f$ is arbitrarily large. The contradiction establishes the theorem.

## References

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