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PSEUDO-UNIFORM CONVEXITY IN H^1

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Although the familiar Banach space, H^1, is not uniformly convex, we will display two important properties which H^1 shares with the uniformly convex spaces.

Loosely speaking, these two properties are

1. "Weak convergence + convergence of norms ⇒ convergence" and
2. "A certain kind of minimizing sequence automatically converges."

We delay the precise statements and take this opportunity to point out that these two results were originally conjectured by A. Shields and H. S. Shapiro, both of whom also supplied valuable ideas and suggestions leading to the solutions.

Definitions and notation. Functions of the class H^1 are denoted by $f, g, \cdots$. The H^p norm of f $(1 \leqq p \leqq \infty)$ is written $\|f\|_p$ and we abbreviate $\|f\|_1$ to $\|f\|$. By "weak" convergence of f_n to f, written $f_n \to^w f$, we mean that $f_n(Z)$ converges uniformly to $f(Z)$ in every compact subset of $|Z| < 1$. Ordinary convergence, written $f_n \to f$ means of course that $\|f_n - f\| \to 0$.

Let $F(\theta)$ be any function of class L^1 $(0, 2\pi)$, and consider the problem of minimizing $\int_0^{2\pi} |F(\theta) - f(e^{i\theta})| d\theta$ over all $f \in H^1$. It is a theorem of Rogosinski and Shapiro (see [1]) that this problem has a unique solution for each F, and this solution we denote by f_F. We also denote $|F(\theta) - f_F(e^{i\theta})| = P_F(\theta)$ and $\int_0^{2\pi} P_F(\theta) d\theta = d_F$.

With these definitions, then, we are able to give our precise statements.

THEOREM 1. *If $f_n \to^w f$ and $\|f_n\| \to \|f\|$ then $f_n \to f$.*

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THEOREM 2. *If* $\int_0^{2\pi} |F(\theta) - f_n(e^{i\theta})| d\theta \to d_F$ *then* $f_n \to f_F$.

PROOF OF THEOREM 1. Suppose that the result were false, i.e., that for some subsequence S_1 of the positive integers, and some ϵ, we have

1. $\|f - f_n\| > \epsilon > 0$ for $n \in S_1$.

An immediate corollary to this is

2. f is not identically 0.

Since $\|f_n\| \to \|f\| \neq 0$, then, f_n is not identically 0 for n large enough and we can write $f_n = g_n B_n$ where g_n is zero-free and B_n is a Blaschke product and we obtain immediately

3. $\|f_n\| = \|g_n\|$, $\|B_n\|_\infty = 1$.

Thus $\|g_n\|$ and $\|B_n\|_\infty$ are bounded and so we can take a subsequence $S_2 \subset S_1$ through which both g_n and B_n converge weakly. Suppose then, that,

4. $g_n \to^w g$, $B_n \to^w B$ for $n \in S_2$.

It follows that $f_n = g_n B_n \to^w gB$ so that $f = gB$. On the other hand $\|g\| \leq \limsup \|g_n\| = \limsup \|f_n\| = \|f\|$ while $\|B\|_\infty \leq \limsup \|B_n\|_\infty = 1$. Thus since $\|f\| \leq \|g\| \, \|B\|_\infty$ it follows first that $\|g\| = \|f\|$ and therefore that $|B(e^{i\theta})| = 1$ almost everywhere. Summing up, then, we have

5. $\|g\| = \|f\|$, $|B(e^{i\theta})| = 1$, a.e., $\|g_n\| \to \|g\|$ for $n \in S_2$. g is furthermore zero-free by Hurwicz's theorem.

Now let us examine the Hilbert space H^2 and note that the functions $\sqrt{g_n}$, $\sqrt{g}$ belong to this space. From 5. we conclude that $\|\sqrt{g_n}\|_2 \to \|\sqrt{g}\|_2$ and this combined with 4. ($g_n \to^w g$) implies weak convergence in the Hilbert space sense. Moreover weak convergence together with convergence of norms does imply convergence in a Hilbert space and our conclusion, then, is

6. $\|\sqrt{g_n} - \sqrt{g}\|_2 \to 0$ for $n \in S_2$.

This same analysis on B_n and B which are also members of H^2 yields

7. $\|B_n - B\|_2 \to 0$ for $n \in S_2$.

From 7. we conclude that there is a subsequence $S_3 \subset S_2$ through which $B_n(e^{i\theta}) \to B(e^{i\theta})$, a.e., and this fact tells us, by the dominated convergence theorem, that $\int_0^{2\pi} |g(e^{i\theta})(B_n(e^{i\theta}) - B(e^{i\theta}))| d\theta \to 0$ as $n \to \infty$, $n \in S_3$. In short, we have

8. $\|g(B_n - B)\| \to 0$ for $n \in S_3$.

Next, by an application of the Schwartz inequality we see that $\|g - g_n\| \leq \|\sqrt{g} - \sqrt{g_n}\|_2 \, \|\sqrt{g} + \sqrt{g_n}\|_2$ so that 6., together with the immediate corollary of 6. that $\|\sqrt{g} + \sqrt{g_n}\|_2$ is bounded, tell us that

9. $\|g - g_n\| \to 0$ for $n \in S_2$.

For n restricted to S_3, and large enough, then $\|f_n - f\| = \|g_n B_n - gB\| \leq \|g(B - B_n)\| + \|(g - g_n)B_n\| \leq \|g(B - B_n)\| + \|g - g_n\| < \epsilon$ (by 8. and 9.).

This contradicts 1. and the theorem is proved.

PROOF OF THEOREM 2.

LEMMA. *If* $F\in L^1(0, 2\pi)$ *then there exists a* $U(Z)$ *analytic in* $|Z|<1$ *satisfying there* $U(0)=0$, $|U(Z)|\leqq 1$, *and such that* $U(e^{i\theta})[F(\theta)-f_F(e^{i\theta})]=P_F(\theta)$ *a.e.*

PROOF. See [1, Theorems 6, 8, 11].

The following lemma is perhaps of independent interest.

LEMMA 3. *Let* $P(\theta)\geqq 0$, $\int_0^{2\pi}P(\theta)d\theta=1$. *Given* $\epsilon>0$, $\exists\delta>0$ *such that for all* $f(Z)\in H^1$ *with* $f(0)=0$, $\int|P+f|<1+\delta$ *implies* $\int|f|<\epsilon$.

PROOF. Assume otherwise that P, f_n satisfy the above hypotheses and that $\int|P+f_n|\to 1$ while

1. $\int|f_n|>\epsilon>0$.

Write $f_n=U_n+iV_n$. From the elementary inequality $\sqrt{(a^2+b^2)}\geqq|a|+b^2/(2\sqrt{(a^2+b^2)})$, it follows that

$$2.\qquad \int|P+f_n|\geqq\int|P+U_n|+\frac{1}{2}\int\frac{V_n^2}{|P+f_n|}.$$

Next, by Schwartz' inequality, we obtain

$$3.\qquad \int\frac{V_n^2}{|P+f_n|}\geqq\frac{\left(\int|V_n|\right)^2}{\int|P+f_n|}.$$

And combining 2. and 3. gives

4. $(\int|V_n|)^2\leqq 2\int|P+f_n|\,[\int|P+f_n|-\int|P+U_n|]$

but note that

$$\int|P+U_n|\geqq\int(P+U_n)=1+\int U_n=1+2\pi\operatorname{Re}f_n(0)=1$$

so that 4. becomes

5. $(\int|V_n|)^2\leqq 2\int|P+f_n|\,[\int|P+f_n|-1]$.

Following Zygmund (see [2]), we have

6. $\int|f_n|^{1/2}\leqq 4\sqrt{\pi}(\int|V_n|)^{1/2}$.

Now $\int|V_n|\to 0$ by 5. and so 6. tells us that $\int|f_n|^{1/2}\to 0$. As a result we can extract a subsequence, S, of the integers such that

7. $f_n(e^{i\theta})\to 0$ a.e. for $n\in S$.

Next we use the elementary inequality $|\alpha+\beta|\geqq|\alpha|+|\beta|-2\min(|\alpha|,|\beta|)$ to deduce that $\int|P+f_n|\geqq\int P+\int|f_n|$

$-2\int \min(P, |f_n|)$ or

8. $\int|f_n| \leqq [\int|P+f_n|-1]+\int \min(P, |f_n|)$.

Now let $n\in S$ be sufficiently large. By hypothesis $\int|P+f_n|-1$ can be made $<\epsilon/2$ and by the dominated convergence theorem and 7., so can $\int \min(P, |f_n|)$. Thus 8. gives $\int|f_n|<\epsilon$ and this contradicts **1.** thereby proving the lemma.

The proof of Theorem 2 is now quite simple. If $d_F=0$ there is nothing to prove and so we may assume that $d_F=1$. Now let $U(Z)$ have the meaning given by our lemma, and let $\epsilon>0$ be given. Next let $\delta>0$ be so small that $\delta<\epsilon/3$ and that $\int|P_F+Ug|<1+\delta\Rightarrow\int|Ug|<\epsilon/3$ all for $g\in H^1$. (This δ can be so chosen by Lemma 3, since $U(0)=0$.)

Assume, then, that $\int|F-f_n|\rightarrow 1$. For n large enough, we have $1+\delta>\int|F-f_n|\geqq\int|P_F+U(f_F-f_n)|$ and so we may conclude that $\int|U(f_F-f_n)|<\epsilon/3$.

If we denote by S the set on $|z|=1$ where $P_F\neq 0$ then of course $|U|=1$ on S and so further conclude that

1. $\int_S|f_F-f_n|<\epsilon/3$.

It next follows that

$$\int_S |F-f_n| \geqq \int_S |F-f_F| - \int_S |f_F-f_n|$$

$$= \int |F-f_F| - \int_S |f_F-f_n| \geqq 1-\epsilon/3$$

and so, since $\int|F-f_n|<1+\delta$, we conclude that $\int_{\tilde{S}}|F-f_n|<\delta+\epsilon/3 <2\epsilon/3$ but $F=f_F$ a.e. in $\tilde{S}$ so that this can be written:

2. $\int_{\tilde{S}}|f_F-f_n|<2\epsilon/3$.

Adding 1. and 2. finally gives, for n sufficiently large, $\int|f_F-f_n|<\epsilon$ and this, of course, is exactly what had to be proved.

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