EXAMPLE OF A NONACYCLIC CONTINUUM SEMIGROUP S WITH ZERO AND $S=ESE$

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Throughout this discussion S will denote a compact connected topological semigroup and E will denote the set of idempotents of S. The problem to be considered concerns a question posed by Professor A. D. Wallace. In [1], Wallace proves that if S has a left unit, if I is a closed ideal of S, and if $L=\square$ or if L is a closed left ideal of S, then $H^n(S)\cong H^n(I\cup L)$ for all integers n, where $H^n(A)$ denotes the nth Alexander-Čech cohomology group of A with coefficients in an arbitrary but fixed group G. If S is assumed to have both a left zero and a left unit, then it follows that each closed left ideal L, of S is acyclic; that is, $H^p(L)=0$ for all $p\geqq 1$. A dual statement holds for closed right ideals if S has a right unit and right zero. A generalization of the case in which S has a left, right, or two-sided unit, is to require that $S=ES$, $S=SE$, or $S=ESE$, respectively, and Wallace has asked: "If S has a zero, are closed right or left ideals of S necessarily acyclic in the more general situation?" [3]. A negative answer to this question is given here by way of examples, and a theorem is proved giving a necessary and sufficient condition for closed right ideals of S to be acyclic, assuming $S=ESE$ and S has a zero. Following the proof of this theorem is an example of a semigroup not satisfying this condition.

The above-mentioned example shows that even though S is acyclic, it is not necessarily true that all closed right ideals of S are acyclic. Thus the question remains as to whether S is acyclic if $S=ESE$ and S has a zero [3]. Wallace proves in [2] that for such a semigroup S, $H^1(S)=0$, however, an example is given here of a semigroup S with zero, $S=ESE$ and $H^2(S)\cong G$ for all groups G, showing that this question also has a negative answer. Two further examples are included in this paper which show what can occur if one only assumes that $S=SE$, or $S=ES$. One example is of a semigroup S with zero, $S=SE$ and $H^1(S)\cong G$ for all groups G and the other is an example of a semigroup S with zero and left unit and S contains a closed right ideal R with $H^1(R)\cong G$.

DEFINITION. Let T be a semigroup, $a\in T$ and $R(a)$ the closed right ideal of T generated by a. Then a is said to be right codependent on

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T if for any integer $n \geqq 1$, $H^n(T)=0$ implies that $H^n(R(a))=0$.

THEOREM. *Let S be a compact connected semigroup with zero and $S=ESE$. A necessary and sufficient condition that each closed right ideal of S be acyclic is that each a in S be right codependent on S.*

The proof of this theorem depends on the following two lemmas. The proofs of these lemmas are paraphrases of the proof of the main theorem in [2] and will be omitted.

LEMMA 1. *Let S be a compact connected semigroup with zero and $S=ES$. Let n be a fixed integer $n \geqq 2$. If $H^{n-1}(R)=0$ for each closed right ideal $R \subset S$, then $H^n(S)=0$.*

LEMMA 2. *Let S be a compact connected semigroup with zero and $S=SE$. Let n be a fixed integer, $n \geqq 1$. If for each $a \in S$, $H^n(aS)=0$ and if for each closed subset $A \subset S$, $H^{n-1}(AS)=0$, then $H^n(R)=0$ for each closed right ideal $R \subset S$. (For $n=1$, reduced groups are to be used.)*

PROOF OF THEOREM. First assume that each a in S is right codependent on S. The proof of sufficiency will be by induction on n. Let $n=1$. From [2], $H^1(S)=0$, hence it follows that $H^1(aS)=0$ for each a in S since $R(a)=aS$ and each a is right codependent on S. Each closed right ideal $R \subset S$ is connected, therefore $H^0(R, r)=0$ for each $r \in R$. Thus, using reduced groups, it follows from Lemma 2 that $H^1(R)=0$ for each closed right ideal $R \subset S$.

Assume now that $H^{k-1}(R)=0$ for each closed right ideal R of S and integer $k \geqq 2$. Then by Lemma 1, $H^k(S)=0$, hence $H^k(aS)=0$ for each $a \in S$. Applying Lemma 2, it follows that $H^k(R)=0$ where R is a closed right ideal of S. This completes the proof of sufficiency.

If each closed right ideal of S is acyclic, then $H^n(aS)=0$ for each $a \in S$, integer $n \geqq 1$ and coefficient group G since aS is a closed right ideal. Also $R(a)=aS$ for each $a \in S$ so that it is trivially true that each a in S is right codependent on S which completes the proof of the theorem.

In the following examples, let $I=[0, 1]$ denote the real unit interval and for x and y in I let:

$$x \wedge y = \text{minimum of } x \text{ and } y,$$
$$x \vee y = \text{maximum of } x \text{ and } y,$$
$$xy = \text{real product of } x \text{ and } y.$$

EXAMPLE 1. This is an example of a compact connected semigroup S with zero, $S=ESE$, S is acyclic and there exists an element p in S with $H^1(pS) \cong G$ for all groups G. This example shows that there exist semigroups with zero such that each element is not right codependent on S. The topological space of S is a two-cell with three closed intervals, I_1, I_2, and I_3, issuing from a common point z_0, on the boundary, B, of the two-cell. This point z_0 is the zero of S and $p \in B \backslash z_0$. By construction $pS = B$ and $B^2 = z_0$. In this example, $E = \{e_1, e_2, e_3, z_0\}$ where e_i is the free endpoint of I_i.

Example 1 is constructed as follows. Let $\{a, b, c, d, \theta\}$ be a discrete space consisting of five elements. Define spaces A, B, C, D and S_0 as follows:

$A = a \times I$, $B = b \times I$, $C = c \times I$, $D = d \times I \times I$, each with the product topology and $S_0 = A \cup B \cup C \cup D \cup \{\theta\}$ with the topology on S_0 given by the union of the topologies on A, B, C, D and $\{\theta\}$. Define the product pq for p and q in S_0 by:

$$pq = \begin{cases} (d, (x \wedge y)r, (x \vee y)), & \text{if } p = (d, x, y) \in D,\ q = (a, r) \in A, \\ (d, (x \vee y), (x \wedge y)r), & \text{if } p = (d, x, y) \in D,\ q = (b, r) \in B, \\ (b, rs), & \text{if } p = (a, r) \in A, \text{ or } p = (b, r) \in B \text{ and } q = (b, s) \in B, \\ (a, rs), & \text{if } p = (b, r) \in B, \text{ or } p = (a, r) \in A \text{ and } q = (a, s) \in A, \\ (d, xr, yr), & \text{if } p = (c, r) \in C \text{ and } q = (d, x, y) \in D, \\ (c, rs), & \text{if } p = (c, r) \in C \text{ and } q = (c, s) \in C, \\ \theta & \text{otherwise.} \end{cases}$$

By the definition of the topology on S_0, multiplication is continuous and associativity is checked by direct computation. Let $E_0 = \{(a, 1), (b, 1), (c, 1), \theta\}$. Then E_0 is a set of idempotents in S_0 and the claim is made that $S_0 = E_0 S_0 E_0$. This is true since $(a, 1)$ is a two-sided unit for A and a right unit for $(d \times \{(x, y) : x \leqq y\})$; $(b, 1)$ is a two-sided unit for B and a right unit for $(d \times \{(x, y) : x \geqq y\})$; $(c, 1)$ is a two-sided unit for C and a left unit for D; and $\theta^2 = \theta$.

Consider now, $I_0 = (d \times \{0\} \times I) \cup (d \times I \times \{0\}) \cup \{(a, 0), (b, 0), (c, 0), \theta\}$. By direct computation it can be shown that this closed subset of S_0 is a two-sided ideal of S_0. Let $S = S_0/I_0$ be the Rees quotient of S_0 by I_0. Then S is a compact connected semigroup with zero and it is clear that S is acyclic. Also the condition $S_0 = E_0 S_0 E_0$ implies that $S = ESE$ where E is the set of idempotents of S.

Let $p = (d, 1, 1)$. Then $pS = ((d, 1, 1)S_0 \cup I_0)/I_0 = ((d \times I \times \{1\}) \cup (d \times \{1\} \times I) \cup I_0)/I_0$ so that pS is homeomorphic to a one-sphere and therefore $H^1(pS) \cong G$ for all groups G.

EXAMPLE 2. This is an example of a compact connected semigroup S with zero, $S = ESE$ and $H^2(S) \cong G$ for all coefficient groups G. The topological space of this semigroup is a two-sphere with four closed intervals, I_i, $i = 1, 2, 3, 4$, issuing from a common point, z_1, on the two-sphere. The point z_1 is a zero for S and if e_i denotes the free endpoint of I_i, then $E=\{e_1, e_2, e_3, e_4, z_1\}$ and multiplication in S has the following properties: (Let S_1 denote the two-sphere in S; C_1 and C_2 the two great circles in S_1 through z_1; H_1 and H_2 the closed hemispheres determined by C_1; and P_1, P_2 the closed hemispheres determined by C_2.) $S_1^2=z_1$; $e_1S=H_1\cup I_1$; $e_2S=H_2\cup I_2$; $Se_3=P_1\cup I_3$; $Se_4=P_2\cup I_4$. Hence $e_1S\cap e_2S=C_1$ is a closed right ideal of S with nontrivial cohomology in dimension one. Similarly, $Se_3\cap Se_4=C_2$ is a closed left ideal of S.

S is constructed in the following way: Let $N=\{a, b, c, d, e, \theta\}$ be a discrete space with six elements. Let $N_0=N\backslash\{e, \theta\}$ and let $T=(N_0\times I)\cup(e\times I\times I)\cup\{\theta\}$ with the topology on T given by the union of the topologies of its subsets. For x and y in $[0, 1]$ define $\alpha(x, y)=(x\wedge y\wedge(1-x)\wedge(1-y))$ and define the product pq for p and q in T by:

$$pq = \begin{cases} (e, (x\wedge(1-y))-r\alpha(x, y), (y\wedge(1-x))-r\alpha(x, y)), & p=(a, r)\in A, \\ & q=(e, x, y)\in S_0, =e\times I\times I \\ (e, (x\vee(1-y))+r\alpha(x, y), (y\vee(1-x))+r\alpha(x, y)), & p=(b, r)\in B, \\ & q=(e, x, y)\in S_0 \\ (a, r+s-rs), & p=(a, r)\in A \text{ and } q=(a, s)\in A \text{ or } q=(b, s)\in B, \\ (b, r+s-rs), & p=(b, r)\in B \text{ and } q=(a, s)\in A \text{ or } q=(b, s)\in B, \\ (e, 1-s+(x\vee y)s, (x\wedge y)s), & p=(e, x, y)\in S_0,\ q=(c, s)\in C, \\ (e, (x\wedge y)s, 1-s+(x\vee y)s), & p=(e, x, y)\in S_0,\ q=(d, s)\in D, \\ (d, rs), & p=(c, r)\in C \text{ or } p=(d, r)\in D \text{ and } q=(d, s)\in D, \\ (c, rs), & p=(d, r)\in D \text{ or } p=(c, r)\in C \text{ and } q=(c, s)\in C, \\ \theta & \text{otherwise.} \end{cases}$$

By the definition of the topology on T, it is clear that multiplication is continuous since it involves continuous operations of real numbers. By direct computation it is seen that multiplication is also associative and therefore T is a compact semigroup. Let S_i, $i=1, 2, 3, 4$, subsets of $e\times I\times I$ be defined by:

$$S_1 = \{(e, x, y): 0 \leqq x \leqq y \leqq x + y \leqq 1\},$$
$$S_2 = \{(e, x, y): 0 \leqq x \leqq y \leqq 1 \leqq x + y\},$$
$$S_3 = \{(e, x, y): 0 \leqq y \leqq x \leqq 1 \leqq x + y\},$$
$$S_4 = \{(e, x, y): 0 \leqq y \leqq x \leqq x + y \leqq 1\},$$

and let $E_0=\{(a, 0), (b, 0), (c, 1), (d, 1), \theta\}$. E_0 is a set of idempotents in T and the claim is made that $T=E_0TE_0$. This follows from the following equalities:

$a \times I = (a, 0)(a \times I)(a, 0)$; $b \times I = (b, 0)(b \times I)(b, 0)$; $c \times I = (c, 1)(c\times I)(c, 1)$; $d\times I=(d, 1)(d\times I)(d, 1)$; $\theta^2=\theta$ and $e\times I\times I = \bigcup\{S_i: i=1, 2, 3, 4\} = (a, 0)S_1(d, 1)\cup(b, 0)S_2(d, 1)\cup(b, 0)S_3(c, 1)\cup(a, 0)S_4(c, 1)$. This proves that $T=E_0TE_0$ as claimed.

Now let $I_0=\{(a, 1), (b, 1), (c, 0), (d, 0), \theta\}\cup(e\times F(I\times I))$ where $F(I\times I)$ denotes the boundary of $I\times I$ in the Euclidean plane. It can be shown that this closed subset of T is a two-sided ideal of T, hence $S=T/I_0$ is a compact connected semigroup as described above. Also $S=ESE$, since $T=E_0TE_0$ and S has a zero.

EXAMPLE 3. This example is of a semigroup $S=SE$ which is compact connected, has a zero and $H^1(S)\cong G$ for all groups G. S is a subsemigroup of the semigroup in Example 1 and the topological space of S is a circle with two closed intervals issuing from a common point of the circle.

In the terminology of Example 1, consider the following closed subsemigroup, T, of S_0:

$T=A\cup B\cup(d\times F(I\times I))\cup\{\theta\}$. Then $T=T(a, 1)\cup T(b, 1)\cup\{\theta\}$ so that $T=TE_1$ where E_1 is the set of idempotents in T. Let $I_1 = \{(a, 0), (b, 0), \theta\}\cup(d\times\{0\}\times I)\cup(d\times I\times\{0\})$. Then $I_1=T\cap I_0$ is a closed ideal of T and $S=T/I_1$ is a compact connected semigroup with zero and $S=SE$. Clearly S is topologically as described above, so that $H^1(S)\cong G$.

EXAMPLE 4. This final example is of a semigroup S with zero and left unit and S contains a closed right ideal R such that $H^1(R)\cong G$, for all groups G.

Let $S=(\{0\}\times I\times I)\cup(I\times\{0\}\times I)$ and define multiplication in S by $(x, y, z)(r, s, t)=(xr, xs, zt)$. S can be represented by the following matrix semigroup:

$$\left\{\begin{pmatrix} x & y & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} : (x, y, z) \in S\right\}$$

so that multiplication in S is continuous and associative. Clearly $(0, 0, 0)$ is a zero for S and $(1, 0, 1)$ is a left unit for S. By the definition of multiplication it follows that any subset of $(\{0\}\times I\times I)$ containing $(\{0\}\times\{0\}\times I)$ is a right ideal of S and, in particular, $R=$the boundary of $(\{0\}\times I\times I)$ is a closed right ideal of S and $H^1(R)\cong G$ for all coefficient groups G.

In these four examples it might be noted that the set of idempotents in each semigroup was a finite discrete set. It might be of interest to know if there exists a semigroup $S=ESE$ which is compact connected, has a zero, is not acyclic and such that the set of idempotents is connected.

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CONFORMAL VECTOR FIELDS IN COMPACT RIEMANNIAN MANIFOLDS

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1. **Introduction.** Let V^n be a compact Riemannian manifold of dimension n and of class C^3. Let $g_{ij}(x)$ of class C^2 be the coefficients of the fundamental metric which is assumed to be positive definite. Let Γ^h_{ij} be the Christoffell symbol, R_{ijhk} the curvature tensor and R_{ij} the Ricci tensor.

Let ϕ be an arbitrary scalar invariant, ξ^i an arbitrary vector field and $\xi_{i_1i_2\cdots i_p}$ an arbitrary anti-symmetric tensor field of order p, all of class C^2 in V^n. We shall make use of the following results obtained by S. Bochner and K. Yano [1, pp. 31, 51, 69]:

(1.1) $(\Delta\phi\geqq 0$ everywhere in $V^n)\Rightarrow(\phi=\text{constant}$ everywhere in $V^n)$.

$$\int_{V^n}\xi^i{}_{,i}\,dv = 0. \tag{1.2}$$

$$\int_{V^n}(R_{ij}\xi^i\xi^j + \xi^i{}_{,j}\,\xi^j{}_{,i} - \xi^i{}_{,i}\,\xi^j{}_{,j})dv = 0. \tag{1.3}$$

$$\int_{V^n}(F\{\xi_{i_1i_2\cdots i_p}\} + \xi^{ii_2\cdots i_p,j}\,\xi_{ji_2\cdots i_p,i} - \xi^{ii_2\cdots i_p}{}_{,i}\,\xi^j{}_{i_2\cdots i_p,j})dv=0 \tag{1.4}$$

where

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