EXAMPLE OF A NONACYCLIC CONTINUUM SEMIGROUP
S WITH ZERO AND S = ESE

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Throughout this discussion S will denote a compact connected
topological semigroup and E will denote the set of idempotents of S.
The problem to be considered concerns a question posed by Professor
A. D. Wallace. In [1], Wallace proves that if S has a left unit, if I
is a closed ideal of S, and if L = 0 or if L is a closed left ideal of S,
then $H^n(S) \cong H^n(I \cup L)$ for all integers $n$, where $H^n(A)$ denotes the
$n$th Alexander-Čech cohomology group of A with coefficients in an
arbitrary but fixed group G. If S is assumed to have both a left zero
and a left unit, then it follows that each closed left ideal L, of S is
acyclic; that is, $H^n(L) = 0$ for all $p \geq 1$. A dual statement holds for
closed right ideals if S has a right unit and right zero. A generaliza-
tion of the case in which S has a left, right, or two-sided unit, is to
require that $S = ES$, $S = SE$, or $S = ESE$, respectively, and Wallace
has asked: “If S has a zero, are closed right or left ideals of S necessarily
acyclic in the more general situation?” [3]. A negative answer to
this question is given here by way of examples, and a theorem is
proved giving a necessary and sufficient condition for closed right ideals of S to be acyclic, assuming $S = ESE$ and S has a zero. Follow-
ing the proof of this theorem is an example of a semigroup not satisfying
this condition.

The above-mentioned example shows that even though S is acyclic,
it is not necessarily true that all closed right ideals of S are acyclic.
Thus the question remains as to whether S is acyclic if $S = ESE$ and
S has a zero [3]. Wallace proves in [2] that for such a semigroup S,
$H^1(S) = 0$, however, an example is given here of a semigroup S with
zero, $S = ESE$ and $H^2(S) \cong G$ for all groups G, showing that this
question also has a negative answer. Two further examples are in-
cluded in this paper which show what can occur if one only assumes
that $S = SE$, or $S = ES$. One example is of a semigroup S with zero,
$S = SE$ and $H^1(S) \cong G$ for all groups G and the other is an example of a
semigroup S with zero and left unit and S contains a closed right
ideal $R$ with $H^1(R) \cong G$.

Definition. Let T be a semigroup, $a \in T$ and $R(a)$ the closed right
ideal of T generated by a. Then a is said to be right codependent on

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THEOREM. Let $S$ be a compact connected semigroup with zero and $S = ESE$. A necessary and sufficient condition that each closed right ideal of $S$ be acyclic is that each $a$ in $S$ be right codependent on $S$.

The proof of this theorem depends on the following two lemmas. The proofs of these lemmas are paraphrases of the proof of the main theorem in [2] and will be omitted.

**Lemma 1.** Let $S$ be a compact connected semigroup with zero and $S = ESE$. Let $n$ be a fixed integer $n \geq 2$. If $H^{n-1}(R) = 0$ for each closed right ideal $R \subseteq S$, then $H^n(S) = 0$.

**Lemma 2.** Let $S$ be a compact connected semigroup with zero and $S = ESE$. Let $n$ be a fixed integer, $n \geq 1$. If for each $a \in S$, $H^n(aS) = 0$ and if for each closed subset $A \subseteq S$, $H^{n-1}(A, S) = 0$, then $H^n(R) = 0$ for each closed right ideal $R \subseteq S$. (For $n = 1$, reduced groups are to be used.)

**Proof of Theorem.** First assume that each $a$ in $S$ is right codependent on $S$. The proof of sufficiency will be by induction on $n$. Let $n = 1$. From [2], $H^1(S) = 0$, hence it follows that $H^1(aS) = 0$ for each $a$ in $S$ since $R(a) = aS$ and each $a$ is right codependent on $S$. Each closed right ideal $R \subseteq S$ is connected, therefore $H^0(R, r) = 0$ for each $r \in R$. Thus, using reduced groups, it follows from Lemma 2 that $H^1(R) = 0$ for each closed right ideal $R \subseteq S$.

Assume now that $H^{k-1}(R) = 0$ for each closed right ideal $R$ of $S$ and integer $k \geq 2$. Then by Lemma 1, $H^k(S) = 0$, hence $H^k(aS) = 0$ for each $a \in S$. Applying Lemma 2, it follows that $H^k(R) = 0$ where $R$ is a closed right ideal of $S$. This completes the proof of sufficiency.

If each closed right ideal of $S$ is acyclic, then $H^n(aS) = 0$ for each $a \in S$, integer $n \geq 1$ and coefficient group $G$ since $aS$ is a closed right ideal. Also $R(a) = aS$ for each $a \in S$ so that it is trivially true that each $a$ in $S$ is right codependent on $S$ which completes the proof of the theorem.

In the following examples, let $I = [0, 1]$ denote the real unit interval and for $x$ and $y$ in $I$ let:

\[
\begin{align*}
x \land y &= \text{minimum of } x \text{ and } y, \\
x \lor y &= \text{maximum of } x \text{ and } y, \\
x y &= \text{real product of } x \text{ and } y.
\end{align*}
\]

**Example 1.** This is an example of a compact connected semigroup $S$ with zero, $S = ESE$, $S$ is acyclic and there exists an element $p$ in $S$ with $H^1(pS) \cong G$ for all groups $G$. This example shows that there exist
semigroups with zero such that each element is not right codependent on 5. The topological space of 5 is a two-cell with three closed intervals, I₁, I₂, and I₃, issuing from a common point z₀, on the boundary, B, of the two-cell. This point z₀ is the zero of 5 and p ∈ B \ z₀. By construction pS = B and B^2 = z₀. In this example, E = {e₁, e₂, e₃, z₀} where eᵢ is the free endpoint of Iᵢ.

Example 1 is constructed as follows. Let \{a, b, c, d, θ\} be a discrete space consisting of five elements. Define spaces A, B, C, D and S₀ as follows:

\[
A = a \times I, \quad B = b \times I, \quad C = c \times I, \quad D = d \times I \times I,
\]
each with the product topology and \(S₀ = A \cup B \cup C \cup D \cup \{\theta\}\) with the topology on \(S₀\) given by the union of the topologies on \(A, B, C, D\) and \(\{\theta\}\). Define the product pq for p and q in \(S₀\) by:

\[
\begin{align*}
(pq) &= \begin{cases} 
(a, rs), & \text{if } p = (a, r) \in A, \text{ or } p = (b, r) \in B, \\
(b, rs), & \text{if } p = (a, r) \in A, \text{ or } p = (b, r) \in B, \\
(d, rs), & \text{if } p = (a, r) \in A, \text{ or } p = (b, r) \in B, \\
(c, rs), & \text{if } p = (c, r) \in C \text{ and } q = (c, s) \in C, \\
\theta & \text{ otherwise.}
\end{cases}
\end{align*}
\]

By the definition of the topology on \(S₀\), multiplication is continuous and associativity is checked by direct computation. Let \(E₀ = \{(a, 1), (b, 1), (c, 1), (d, 1), \theta\}\). Then \(E₀\) is a set of idempotents in \(S₀\) and the claim is made that \(S₀ = E₀S₀E₀\). This is true since \((a, 1)\) is a two-sided unit for \(A\) and a right unit for \((d \times \{(x, y) : x ≥ y\})\); \((b, 1)\) is a two-sided unit for \(B\) and a right unit for \((d \times \{(x, y) : x ≥ y\})\); \((c, 1)\) is a two-sided unit for \(C\) and a left unit for \(D\); and \(θ² = θ\).

Consider now, \(I₀ = (d \times \{0\} \times I) \cup (d \times I \times \{0\}) \cup \{(a, 0), (b, 0), (c, 0), \theta\}\). By direct computation it can be shown that this closed subset of \(S₀\) is a two-sided ideal of \(S₀\). Let \(S = S₀/I₀\) be the Rees quotient of \(S₀\) by \(I₀\). Then \(S\) is a compact connected semigroup with zero and it is clear that \(S\) is acyclic. Also the condition \(S₀ = E₀S₀E₀\) implies that \(S = ESE\) where \(E\) is the set of idempotents of \(S\).

Let \(p = (d, 1, 1)\). Then \(pS = ((d, 1, 1)S₀ \cup I₀)/I₀ = ((d \times I \times \{1\}) \cup (d \times \{1\} \times I) \cup I₀)/I₀\) so that \(pS\) is homeomorphic to a one-sphere and therefore \(H^1(pS) \cong G\) for all groups \(G\).

Example 2. This is an example of a compact connected semigroup \(S\) with zero, \(S = ESE\) and \(H^2(S) \cong G\) for all coefficient groups \(G\). The topological space of this semigroup is a two-sphere with four closed intervals, \(Iᵢ, i = 1, 2, 3, 4,\) issuing from a common point, \(zᵢ,\) on the two-
sphere. The point $z_x$ is a zero for $S$ and if $e_i$ denotes the free endpoint of $I_i$, then $E = \{e_1, e_2, e_3, e_4, z_x\}$ and multiplication in $S$ has the following properties: (Let $S_i$ denote the two-sphere in $S$; $C_1$ and $C_2$ the two great circles in $S_i$ through $z_x$; $H_1$ and $H_2$ the closed hemispheres determined by $C_1$; and $P_1$, $P_2$ the closed hemispheres determined by $C_2$.) $S_2 = z_x$; $e_1 S = H_1 \cup I_1$; $e_2 S = H_2 \cup I_2$; $S_3 = P_1 \cup I_3$; $S_4 = P_2 \cup I_4$. Hence $e_1 S \cap e_2 S = C_1$ is a closed right ideal of $S$ with nontrivial cohomology in dimension one. Similarly, $S_3 \cap S_4 = C_2$ is a closed left ideal of $S$.

$S$ is constructed in the following way: Let $N = \{a, b, c, d, e, \theta\}$ be a discrete space with six elements. Let $N_0 = N \setminus \{e, \theta\}$ and let $T = (N_0 \times I) \cup (e \times I \times I) \cup \{\theta\}$ with the topology on $T$ given by the union of the topologies of its subsets. For $x$ and $y$ in $[0, 1]$ define $\alpha(x, y) = (x \wedge y \wedge (1-x) \wedge (1-y))$ and define the product $pq$ for $p$ and $q$ in $T$ by:

$$
\begin{align*}
(p, (e, (x \wedge (1-y)) - r \alpha(x, y), (y \wedge (1-x)) - r \alpha(x, y)), \quad p = (a, r) \in A,
q &= (e, x, y) \in S_0, \quad q = e \times I \times I
\end{align*}
$$

$$
\begin{align*}
(p, (e, (x \vee (1-y)) + r \alpha(x, y), (y \vee (1-x)) + r \alpha(x, y)), \quad p = (b, r) \in B,
q &= (e, x, y) \in S_0
\end{align*}
$$

$$
\begin{align*}
(a, r + s - rs), \quad p = (a, r) \in A \text{ and } q = (a, s) \in A \text{ or } q = (b, s) \in B,
(b, r + s - rs), \quad p = (b, r) \in B \text{ and } q = (a, s) \in A \text{ or } q = (b, s) \in B
\end{align*}
$$

$$
\begin{align*}
(e, 1 - s + (x \vee y)s, (x \wedge y)s), \quad p = (e, x, y) \in S_0, \quad q = (c, s) \in C,
(c, 1 - s + (x \vee y)s), \quad p = (e, x, y) \in S_0, \quad q = (d, s) \in D
\end{align*}
$$

$$
\begin{align*}
(d, rs), \quad p = (c, r) \in C \text{ or } p = (d, r) \in D \text{ and } q = (d, s) \in D,
(c, rs), \quad p = (d, r) \in D \text{ or } p = (c, r) \in C \text{ and } q = (c, s) \in C,
\theta \quad \text{otherwise}.
\end{align*}
$$

By the definition of the topology on $T$, it is clear that multiplication is continuous since it involves continuous operations of real numbers. By direct computation it is seen that multiplication is also associative and therefore $T$ is a compact semigroup. Let $S_i, i = 1, 2, 3, 4$, subsets of $e \times I \times I$ be defined by:

$$
S_1 = \{(e, x, y) : 0 \leq x \leq y \leq x + y \leq 1\},
$$

$$
S_2 = \{(e, x, y) : 0 \leq x \leq y \leq 1 \leq x + y\},
$$

$$
S_3 = \{(e, x, y) : 0 \leq y \leq x \leq 1 \leq x + y\},
$$

$$
S_4 = \{(e, x, y) : 0 \leq y \leq x \leq x + y \leq 1\},
$$

and let $E_0 = \{(a, 0), (b, 0), (c, 1), (d, 1), \theta\}$. $E_0$ is a set of idempotents.
in $T$ and the claim is made that $T = E_0 T E_0$. This follows from the following equalities:
\[
    a \times I = (a, 0)(a \times I)(a, 0); \quad b \times I = (b, 0)(b \times I)(b, 0); \quad c \times I = (c, 1)(c \times I)(c, 1); \quad d \times I = (d, 1)(d \times I)(d, 1); \quad \theta^2 = \theta \quad \text{and} \quad e \times I \times I \\
    = \bigcup_{i=1, 2, 3, 4} d_0(d, 1) \bigcup (b, 0) d_0(d, 1) \bigcup (b, 0) d_0(c, 1) \\
    \cup (a, 0) d_0(c, 1). \quad \text{This proves that} \quad T = E_0 T E_0 \text{ as claimed.}
\]

Now let $I_0 = \{(a, 1), (b, 1), (c, 0), (d, 0), \theta\} \cup (e \times F(I \times I))$ where $F(I \times I)$ denotes the boundary of $I \times I$ in the Euclidean plane. It can be shown that this closed subset of $T$ is a two-sided ideal of $T$, hence $S = T / I_0$ is a compact connected semigroup as described above. Also $S = E_0 S E_0$, since $T = E_0 T E_0$ and $S$ has a zero.

**Example 3.** This example is of a semigroup $S = SE$ which is compact connected, has a zero and $H^1(S) \cong G$ for all groups $G$. $S$ is a subsemigroup of the semigroup in Example 1 and the topological space of $S$ is a circle with two closed intervals issuing from a common point of the circle.

In the terminology of Example 1, consider the following closed subsemigroup, $T$, of $S_0$:
\[
    T = A \cup B \cup (d \times F(I \times I)) \cup \{\theta\}. \quad \text{Then} \quad T = T(a, 1) \cup T(b, 1) \cup \{\theta\} \quad \text{so that} \quad T = TE_1 \text{ where } E_1 \text{ is the set of idempotents in } T. \quad \text{Let } I_1 = \{(a, 0), (b, 0), \theta\} \cup (d \times \{0\} \times I) \cup (d \times I \times \{0\}). \quad \text{Then } I_1 = T \cap I_0 \text{ is a closed ideal of } T \text{ and } S = T / I_1 \text{ is a compact connected semigroup with zero and } S = SE. \quad \text{Clearly } S \text{ is topologically as described above, so that } H^1(S) \cong G.
\]

**Example 4.** This final example is of a semigroup $S$ with zero and left unit and $S$ contains a closed right ideal $R$ such that $H^1(R) \cong G$, for all groups $G$.

Let $S = \{0\} \times I \times I \cup (I \times \{0\} \times I)$ and define multiplication in $S$ by $(x, y, z)(r, s, t) = (xr, xs, zt)$. $S$ can be represented by the following matrix semigroup:
\[
    \begin{bmatrix}
        x & y & 0 \\
        0 & 0 & 0 \\
        0 & 0 & z
    \end{bmatrix} : (x, y, z) \in S
\]

so that multiplication in $S$ is continuous and associative. Clearly $(0, 0, 0)$ is a zero for $S$ and $(1, 0, 1)$ is a left unit for $S$. By the definition of multiplication it follows that any subset of $\{0\} \times I \times I$ containing $\{0\} \times \{0\} \times I$ is a right ideal of $S$ and, in particular, $R$ - the boundary of $(\{0\} \times I \times I)$ is a closed right ideal of $S$ and $H^1(R) \cong G$ for all coefficient groups $G$.

In these four examples it might be noted that the set of idem-
potents in each semigroup was a finite discrete set. It might be of interest to know if there exists a semigroup $S = ESE$ which is compact connected, has a zero, is not acyclic and such that the set of idempotents is connected.

Bibliography


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CONFORMAL VECTOR FIELDS IN COMPACT RIEMANNIAN MANIFOLDS

T. K. Pan

1. Introduction. Let $V^n$ be a compact Riemannian manifold of dimension $n$ and of class $C^2$. Let $g_{ij}(x)$ of class $C^2$ be the coefficients of the fundamental metric which is assumed to be positive definite. Let $\Gamma^i_{jk}$ be the Christoffell symbol, $R_{ijkl}$ the curvature tensor and $R_{ij}$ the Ricci tensor.

Let $\phi$ be an arbitrary scalar invariant, $\xi^i$ an arbitrary vector field and $\xi^{i_1i_2\ldots i_p}$ an arbitrary anti-symmetric tensor field of order $p$, all of class $C^2$ in $V^n$. We shall make use of the following results obtained by S. Bochner and K. Yano [1, pp. 31, 51, 69]:

(1.1) $\Delta \phi \geq 0$ everywhere in $V^n \Rightarrow \phi = \text{constant everywhere in } V^n$.

(1.2) $\int_{V^n} \xi^i \, dv = 0$.

(1.3) $\int_{V^n} (R_{ij} \xi^i \xi^j + \xi^i \xi^j \xi^i - \xi^i \xi^i \xi^j) \, dv = 0$.

(1.4) $\int_{V^n} (F[\xi^{i_1i_2\ldots i_p}] + \xi^{i_1i_2\ldots i_p} f \xi_{i_1i_2\ldots i_p} \xi^i - \xi^{i_1i_2\ldots i_p} \xi_{i_1i_2\ldots i_p} \xi^i) \, dv = 0$

where