ON THE PRODUCT OF DIRECTED GRAPHS

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1. **Introduction.** The concept of the product of two graphs [**1**, p. 23] has application in the theory of games and the theory of automata. Weichsel [**2**] deduced a necessary and sufficient condition for the product of two connected undirected graphs to be connected; he further proves that the product graph, if not connected, has exactly two components. In this paper we deduce the corresponding results for directed graphs, namely that if $G_1, G_2, \cdots, G_r$ are strongly connected directed graphs then $G_1 \times G_2 \times \cdots \times G_r$ has exactly

$$\frac{d(G_1) \cdot d(G_2) \cdots d(G_r)}{\operatorname{lcm}(d(G_1), d(G_2), \cdots, d(G_r))}$$

components where $d(G)$ is as defined below.

2. **Definitions.** We define a *directed graph* G to be a set of nodes $N(G)$ and a set of edges $E(G)$, the elements of $E(G)$ being ordered pairs (p, q) with $p, q \in N$.

We write $p \Rightarrow q$ if $(p, q) \in E$.

We define a *path* $\mathcal{P}$ of G as a finite sequence $\{p_i\}$, $(0 \leqq i \leqq n, n \geqq 1)$ such that $p_i \Rightarrow p_{i+1}$, $(0 \leqq i \leqq n-1)$. We define $n(\mathcal{P}) = n$, the number of edges in $\mathcal{P}$ and write $\mathcal{P}(p, q)$ for a path with $p_0 = p$, $p_n = q$.

By $p \in \mathcal{P}$ we mean $\mathcal{P}$ is the path $\{p_i\}$ and for some i, $p_i = p$.

We write $p \rightarrow q$ if either $p = q$ or there is a path $\mathcal{P}(p, q)$.

We say p, q are *strongly connected* if $p \rightarrow q$ and $q \rightarrow p$. This is clearly an equivalence relation; we define a component C of G to be a subgraph whose nodes, $N(C)$, are just the nodes in an equivalence class with respect to this relation and whose edges are those edges (p, q) of G with $p, q \in N(C)$.

We say G is *strongly connected* if each pair of nodes of G are strongly connected.

If $\mathcal{P}_1(p_1, p_2)$ and $\mathcal{P}_2(p_2, p_3)$ are paths of G, we define $\mathcal{P}_1 + \mathcal{P}_2$ to be the path $\mathcal{P}_3(p_1, p_3)$ obtained by concatenating the sequences of nodes in $\mathcal{P}_1$ and $\mathcal{P}_2$. This sum is clearly associative and $n(\mathcal{P}_3) = n(\mathcal{P}_1) + n(\mathcal{P}_2)$.

We define a cycle $\mathcal{C}$ to be a path $\mathcal{P}(p, p)$.

We define $d(G) = \gcd(n(\mathcal{C}))$, taken over all cycles of G.

We define the product $G \times H$ of two graphs G, H as follows

(i) $N(G \times H) = \{(p, q) \mid p \in N(G) \text{ and } q \in N(H)\}$,

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(ii) $E(G\times H)=\{((p_1, q_1), (p_2, q_2)) \mid (p_1, p_2)\in E(G)$ and $(q_1, q_2) \in E(H)\}$.

It may be readily seen that the adjacency matrix of $G\times H$ is just the Kronecker product of the adjacency matrices of G and H. Since the product is clearly associative we may define unambiguously $G_1\times G_2\times \cdots \times G_r$.

3. Preliminary lemmas.

LEMMA 1. *Let* $\mathcal{P}_1(p, q)$, $\mathcal{P}_2(p, q)$ *be paths of a strongly connected graph* G. *Then*

$$d(G) \mid (n(\mathcal{P}_1) - n(\mathcal{P}_2)).$$

PROOF. Since G is strongly connected there is a path $\mathcal{P}_3(q, p)$. Then $\mathcal{P}_1+\mathcal{P}_3$ and $\mathcal{P}_2+\mathcal{P}_3$ are cycles; hence $d(G) \mid n(\mathcal{P}_i+\mathcal{P}_3)=n(\mathcal{P}_i)+n(\mathcal{P}_3)$, $(i=1, 2)$. The result of the lemma now follows.

Let G be a strongly connected graph and let Z_d be the ring of residue classes mod $d=d(G)$. We may define a function f: $N(G)\to Z_d$ as follows: choose some node p_0 of G. Then for any $p\in N(G)$ define $f(p)=n(\mathcal{P}(p_0, p))$ for any path $\mathcal{P}(p_0, p)$. Since G is strongly connected there is some such path and Lemma 1 ensures that $f(p)$ is single valued.

LEMMA 2. *Let* G *be a strongly connected graph and* f *be defined as above. Then for any path* $\mathcal{P}(p, q)$,

$$n(\mathcal{P}) \equiv f(p) - f(q) \pmod{d(G)}.$$

PROOF. Let $\mathcal{P}_1(p_0, p)$ be a path and let $\mathcal{P}_2(p_0, q)=\mathcal{P}_1(p_0, p)+\mathcal{P}(p, q)$. By definition of f,

$$n(\mathcal{P}_1) \equiv f(p) \pmod{d(G)}$$

and

$$n(\mathcal{P}_2) \equiv f(q) \pmod{d(G)}.$$

The result of the lemma now follows since $n(\mathcal{P}_2)=n(\mathcal{P})+n(\mathcal{P}_1)$.

LEMMA 3. *Let* G *be a strongly connected graph and let* $d=d(G)$. *Then there exists a finite set of cycles* $\mathcal{C}_1, \cdots, \mathcal{C}_t$ *such that*

$$\gcd_{1\leq i\leq t} (n(\mathcal{C}_i)) = d.$$

PROOF. We may construct the set $\mathcal{C}_1 \cdots \mathcal{C}_t$ inductively. Suppose $\mathcal{C}_1, \cdots, \mathcal{C}_r$ already chosen with $\gcd_{1\leq i\leq r} (n(\mathcal{C}_i))=d_r>d$. Then there exists some $\mathcal{C}$ with $d_r \nmid n(\mathcal{C})$. Let this be $\mathcal{C}_{r+1}$; we now have $d_r>d_{r+1}$.

Since $\{d_r\}$ is a strictly monotonically decreasing sequence it must reach d in a finite number of steps.

LEMMA 4. *Let G be a strongly connected graph; let $d=d(G)$. Let $p_1, p_2 \in N(G)$. Then there is an integer n_0 such that for all $n>n_0$ there is a path $\mathcal{P}(p_1, p_2)$ with*

$$n(\mathcal{P}) = nd + f(p_2) - f(p_1). \tag{1}$$

PROOF. Let $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_t$ be a set of cycles with $\gcd_{1\leq i\leq t}(n(\mathcal{C}_i))=d$. The existence of such a set is assured by Lemma 3. Let $n_i=n(\mathcal{C}_i)$. By definition of the gcd, there exist integers $u_1, u_2, \cdots, u_t$ such that

$$\sum_{i=1}^{t} u_i n_i = d. \tag{2}$$

Let $\mathcal{C}_i=\mathcal{P}_i(q_i, q_i)$. Choose some set of paths $\mathcal{P}_0'(p_1, q_1), \mathcal{P}_1'(q_1, q_2), \cdots, \mathcal{P}_t'(q_t, p_2)$. Let $n(\sum_{i=0}^t \mathcal{P}_i') = \sum_{i=0}^t n(\mathcal{P}_i')=K$. Then by Lemma 2,

$$K = ad + f(p_2) - f(p_1) \tag{3}$$

for some integer a.

Let $u=\max_i(|u_i|)$. Let $s=\sum_{i=1}^n n_i/d$; s is integral since $d|n_i$ for every i. We will show that $n_0=a+s+s^2u$ satisfies the requirements of the lemma. Let $n>n_0$. Then for some integral b, c

$$n - a = bs + c \tag{4}$$

with

$$0 \leq c < s. \tag{5}$$

Let

$$v_i = b + cu_i. \tag{6}$$

Then

$$\begin{aligned} v_i &\geq \left(\frac{n-a}{s} - 1\right) - us, && \text{from (4) and (5)} \\ &\geq \frac{n_0 - a}{s} - 1 - us, && \text{since } n > n_0 \\ &= 0 && \text{by definition of } n_0. \end{aligned}$$

I.e., the v_i are non-negative integers. Hence we may construct a path

$$\mathcal{P}(p_1, p_2) = \mathcal{P}_0'(p_1, q_1) + v_1\mathcal{P}_1(q_1, q_1) + \mathcal{P}_1'(q_1, q_2) + \cdots + v_t\mathcal{P}_t(q_t, q_t) + \mathcal{P}_t'(q_t, p_2).$$

Then

$$\begin{aligned} n(\mathcal{P}) &= \sum_{i=0}^{t} n(\mathcal{P}_i') + \sum_{i=1}^{t} v_i n(\mathcal{P}_i) \\ &= K + \sum_{i=1}^{t} v_i n_i \\ &= ad + f(p_2) - f(p_1) + b \sum_{i=1}^{t} n_i + \sum_{i=1}^{t} c u_i n_i, \text{ from (3) and (6)} \\ &= ad + f(p_2) - f(p_1) + bsd + cd, \text{ by (2) and the definition of } s \\ &= nd + f(p_2) - f(p_1). \end{aligned}$$

This completes the proof of the lemma.

4. Main theorems.

THEOREM 1. *Let G and H be strongly connected graphs. Let $d_1 = d(G)$, $d_2 = d(H)$, $d_3 = \gcd(d_1, d_2)$ and $D = \operatorname{lcm}(d_1, d_2)$. Then, in $G \times H$,*

(i) *$(p, q) \to (p', q')$ if and only if*

$$f_G(p) - f_H(q) \equiv f_G(p') - f_H(q') \pmod{d_3}.$$

(ii) *The number of components of $G \times H$ is d_3.*

(iii) *If C is any component of $G \times H$ then $d(C) = D$.*

PROOF. (i) If $(p, q) \to (p', q')$ then there exists a path $\{(p_i, q_i)\}$ with $(p_0, q_0) = (p, q)$ and $(p_n, q_n) = (p', q')$. Then $\{p_i\}$ and $\{q_i\}$ are paths $\mathcal{P}$, $\mathcal{P}'$ of G and H respectively with $n(\mathcal{P}) = n(\mathcal{P}') = n$. Hence, by Lemma 1,

$$n \equiv f_G(p') - f_G(p) \pmod{d_1}$$

and

$$n \equiv f_H(q') - f_H(q) \pmod{d_2}.$$

Therefore, since $d_3 \mid d_1, d_2$,

$$f_G(p') - f_G(p) \equiv f_H(q') - f_H(q) \pmod{d_3}. \tag{7}$$

Conversely, if (7) is satisfied, let n_1 be a value of n_0 satisfying the conclusions of Lemma 4 for G, p, p'; let n_2 be defined similarly for H, q, q'. Let a, b be integers such that $ad_1 - bd_2 = d_3$. Let

$$\frac{f_G(p) - f_H(q) - f_G(p') + f_H(q')}{d_3} = u, \tag{8}$$

which is integral by hypothesis. Choose r so that

$$(au + rd_2) \geqq n_1$$

and

$$(bu + rd_1) \geqq n_2.$$

Then by Lemma 4 there is a path $\mathcal{P}_1(p, p')$ of G of length

$$n(\mathcal{P}_1) = (au + rd_2)d_1 + f_G(p') - f_G(p)$$

and a path $\mathcal{P}_2(q, q')$ of H of length

$$n(\mathcal{P}_2) = (bu + rd_1)d_2 + f_H(q') - f_H(q).$$

Then

$$n(\mathcal{P}_2) - n(\mathcal{P}_1) = (ad_1 - bd_2)u - ud_3, \qquad \text{by (8)}$$
$$= 0.$$

Hence $\mathcal{P}_1 \equiv \{p_i\}$ and $\mathcal{P}_2 \equiv \{q_i\}$ are paths of the same length and so $\{(p_i, q_i)\}$ is a path of $G \times H$ whose end-nodes are just (p, q) and (p', q'), i.e., $(p, q) \rightarrow (p', q')$ in $G \times H$.

(ii) Let $l(p, q) = f_G(p) - f_H(q)$, reduced mod d_3. Then, by (i), nodes (p, q) and (p', q') are strongly connected if and only if $l(p, q) = l(p', q')$. For any residue x (mod d_3) there is certainly some node (p, q) with $l(p, q) = x$. Hence the number of components of $G \times H$ is just d_3, each component consisting of just these nodes with the same value of $l(p, q)$.

(iii) Let (p, q), (p', q') be nodes of $G \times H$ in the same component, C. Let $d_0 = d(C)$. By Lemmas 1 and 4 the lengths of paths $\mathcal{P}(p, p')$ in G consist of all but a finite number of the integers

$$nd_1 + f_G(p') - f_G(p) \qquad \text{for } n > 0. \tag{9}$$

Similarly the lengths of paths $\mathcal{P}(q, q')$ in H consist of all but a finite number of the integers

$$nd_2 + f_H(q') - f_H(q) \qquad \text{for } n > 0. \tag{10}$$

The integers common to the arithmetic series (9) and (10) form an arithmetic series of difference $D = \text{lcm}(d_1, d_2)$. Hence the lengths of paths $\mathcal{P}((p, q), (p', q'))$ in C consist of all but a finite number of the integers of this series; but applying Lemmas 1 and 4 to $G \times H$ the lengths of these paths consist of all but a finite number of the integers

$$nd(C) + f_C(p', q') - f_C(p, q), \qquad (n > 0).$$

Therefore $d(C) = D = \text{lcm}(d_1, d_2)$.

THEOREM 2. *Let $G_1, G_2, \cdots, G_s$ be strongly connected graphs. Let $d_i = d(G_i)$ $(i = 1, 2, \cdots, s)$. Then in $G_1 \times G_2 \times \cdots \times G_s$:*

(i) *The number of components is* $\prod d_i/\text{lcm}(d_i)$.
(ii) *Each component* C *has* $d(C)=\text{lcm}(d_i)$.

PROOF. The results are true trivially for $s=1$. We shall complete the proof by induction. Suppose the results are true for $s=r$; i.e., $H\equiv G_1\times G_2\times\cdots\times G_r$ has m components, each component C having $d(C)=d_H$, where

$$d_H = \operatorname*{lcm}_{1\leq i\leq r}(d_i)$$

and

$$m = \frac{\prod_{i=1}^{r} d_i}{d_H}.$$

In $H\times G_{r+1}$ nodes (p, q) and (p', q') are strongly connected only if p and p' are strongly connected in H. Hence the components of $H\times G_{r+1}$ consist of just the components of $C\times G_{r+1}$ for all components C of H. To each such product $C\times G_{r+1}$ we may apply the results of Theorem 1. Each component, C', of $C\times G_{r+1}$ has

$$\begin{aligned} d(C') &= \text{lcm}(d_H, d_{r+1}) \\ &= \text{lcm}(\text{lcm}(d_1, d_2, \cdots, d_r), d_{r+1}) \\ &= \text{lcm}(d_1, d_2, \cdots, d_{r+1}). \end{aligned}$$

The number of components of $C\times G$ is

$$\gcd(d_H, d_{r+1}) = \frac{d_H\cdot d_{r+1}}{\text{lcm}(d_H, d_{r+1})}.$$

Hence the number of components of $H\times G_{r+1}$

$$\begin{aligned} &= m\frac{d_H\cdot d_{r+1}}{\text{lcm}(d_H, d_{r+1})} \\ &= \frac{\prod_{i=1}^{r} d_i}{d_H}\cdot\frac{d_H\cdot d_{r+1}}{\text{lcm}(d_H, d_{r+1})} \\ &= \frac{\prod_{i=1}^{r+1} d_i}{\text{lcm}(d_1, d_2, \cdots, d_{r+1})}. \end{aligned}$$

Hence the results of the theorem are true for $s=r+1$. The theorem now follows by induction.

We may apply the results of Theorems 1 and 2 to undirected graphs by replacing each edge (p, q) of the undirected graph by a pair of edges (p, q), (q, p) and constructing an equivalent directed graph. For such a graph G, $d(G)$ is 1 or 2 according as G has or has not an odd cycle. Theorem 1 of [1] now follows from Theorem 1 above. From Theorem 2 follows directly

THEOREM 3. *Let $G_1, G_2, \cdots, G_r$ be undirected graphs. If exactly s of these graphs have an odd cycle then m, the number of components of $G_1 \times G_2 \times \cdots \times G_r$ is given by*

$$m = \begin{cases} 1 & s \geqq r - 1 \\ 2^{r-s} & s \leqq r - 1. \end{cases}$$

BIBLIOGRAPHY

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