THE ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF A VOLTERRA EQUATION

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1. Introduction. Concerning the equation

$$x'(t) = -\int_0^t a(t-\tau)g(x(\tau))d\tau \qquad \left(' = \frac{d}{dt}\right), \tag{1.1}$$

we prove

THEOREM 1. *Let $a(t)$ and $g(x)$ satisfy*

$$a(t) \in C[0, \infty),\ (-1)^k a^{(k)}(t) \geqq 0 \qquad (0 < t < \infty; k = 0, 1, 2, 3), \tag{1.2}$$

$$\begin{gathered} g(x) \in C(-\infty, \infty),\ xg(x) > 0\ (x \neq 0), \\ G(x) = \int_0^x g(\xi)d\xi \to \infty \qquad (|x| \to \infty). \end{gathered} \tag{1.3}$$

If $a(t) \not\equiv a(0)$ and if $u(t)$ is any solution of (1.1) *which exists on* $0 \leqq t < \infty$, *then*

$$\lim_{t\to\infty} u^{(j)}(t) = 0 \qquad (j = 0, 1, 2). \tag{1.4}$$

While our interest here is with the behavior of the solutions of (1.1) as $t \to \infty$, the following observations on existence and uniqueness are relevant. Suppose $g(x)$ also satisfies a Lipschitz condition on every interval $|x| \leqq X < \infty$. The usual method of successive approximations, together with certain a priori bounds obtained in the present proof, then easily imply that for each initial value (1.1) has a unique solution and that this solution exists on $0 \leqq t < \infty$. Even under the present hypothesis it follows readily from these a priori bounds and recent results of Nohel [2] that every solution of (1.1) can be continued (though not necessarily uniquely) over $0 \leqq t < \infty$.

Differentiating (1.1) yields

$$x''(t) + a(0)g(x(t)) = -\int_0^t a'(t-\tau)g(x(\tau))d\tau. \tag{1.5}$$

This is justified by Lemmas 3 and 4 below even if $a'(0) = -\infty$. Equation (1.5) may be thought of as a nonlinear oscillator with a heredi-

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tary term. If $a(t) \equiv a(0) > 0$, then it is well known from ordinary differential equations that every solution of (1.5) (and, in particular, the $u(t)$ of Theorem 1) is periodic. Thus the hypothesis $a(t) \not\equiv a(0)$ is essential in Theorem 1.

Equation (1.5) may be compared with

$$x''(t) + kx(t) = \int_{t-L}^{t} F(t-\tau)x(\tau)d\tau, \tag{1.6}$$

where k and L are fixed positive constants and where $F(t)$ is a nonnegative continuously decreasing function, which is discussed by Volterra [4]. Apart from (1.5) being nonlinear and (1.6) linear, an essential difference between them is that in the former the hereditary term depends on the entire past history of $x(t)$ and in the latter only on a fixed interval of the past. Also, in [4] it is assumed that $k - \int_0^L F(\tau)d\tau > 0$. This is technically analogous to assuming throughout, which we do not do, that $a(\infty) > 0$. In spite of these distinctions, the motivation for, but not the explicit form of, the energy (or Lyapunov) function (3.1) below comes from [4].

Equation (1.1) has been considered in [1] in connection with a problem of reactor dynamics. There the hypothesis included complete monotonicity of $a(t)$ on $0 \leqq t < \infty$ (i.e., $k = 0, 1, 2, \cdots$ in (1.2)) and $g(x) \equiv x$. An asymptotic expansion of $u(t)$ as $t \to \infty$ was obtained by means of a Tauberian theorem for the Laplace transform. A physically important nonlinearity (see [1]) is $g(x) = \exp[x] - 1$, which satisfies (1.3).

Consider now the linear case $g(x) \equiv x$. Let $u(t)$ denote the solution of (1.1) such that $u(0) = 1$. To complete the analogy with (1.6), one must look at the behavior of all the solutions of (1.5) and not only at those spanned by $u(t)$. Setting $w(t) = \int_0^t u(\tau)d\tau$ and integrating the equation that results from substituting $u(t)$ into (1.1) yields

$$w'(t) = -\int_0^t a(t-\tau)w(\tau)d\tau + 1. \tag{1.7}$$

It follows by differentiation that $w(t)$ is also a solution of (1.5). In fact, it is clear that if x_0, x_0' are given real numbers, then $x(t) = x_0 u(t) + x_0' w(t)$ is the unique solution of (1.5) such that $x(0) = x_0$, $x'(0) = x_0'$. Concerning $w(t)$ we prove

THEOREM 2. *If* (1.2) *is satisfied,* $a(t) \not\equiv a(0)$, $g(x) \equiv x$, *and*
(i) $a(t) \in L_1(0, \infty)$, *then*

$$\lim_{t \to \infty} w(t) = \left\{ \int_0^\infty a(t)dt \right\}^{-1}, \tag{1.8}$$

(ii) $a(\infty)>0$, *then*

$$\lim_{t\to\infty} w(t) = 0. \tag{1.9}$$

From (i), (ii) one is led to conjecture that (1.9) also holds if $a(\infty)=0$, $a(t)\notin L_1(0,\ \infty)$. We have been unable either to establish this or to find a counterexample.

2. **Preliminaries.** In the proofs we shall use the following elementary lemmas.

LEMMA 1. *If $f''(t)$ exists on $0<t<\infty$ and if*

$$f(t) \geqq 0,\quad f'(t) \leqq 0,\quad f''(t) \geqq -K > -\infty \qquad (0 < t < \infty)$$

for some constant K, then $f'(t)\to 0$ as $t\to\infty$.

PROOF. If $f'(t)\not\to 0$ as $t\to\infty$, then by the hypothesis there exist a $\lambda>0$ and a sequence $\{t_n\}$, where $t_n\to\infty$ as $n\to\infty$, such that $f'(t_n) \leqq -\lambda<0$. Consider the intervals

$$I_n = \left[t_n - \frac{\lambda}{2K},\, t_n\right] \text{ for } n \geqq N, \quad \text{where} \quad t_n - \frac{\lambda}{2K} > 0$$

for $n\geqq N$. By the mean value theorem and the hypothesis

$$f'(t) = f'(t_n) + f''(\theta)(t-t_n) \leqq -\lambda + \frac{\lambda}{2} = -\frac{\lambda}{2} \quad (t\in I_n, n \geqq N),$$

where $t<\theta=\theta(t,\,t_n)<t_n$. A second application of the mean value theorem yields

$$f\left(t_n - \frac{\lambda}{2K}\right) - f(t_n) \geqq \frac{\lambda}{2}\left(\frac{\lambda}{2K}\right) = \frac{\lambda^2}{4K} \qquad (n \geqq N),$$

which clearly contradicts $f(t)\downarrow f(\infty)\geqq 0$ as $t\to\infty$ and thus completes the proof.

It may be similarly shown that the preceding lemma is also true if $f''(t)$ is bounded from above rather than from below.

Although $a(\infty)\geqq 0$ is not necessarily zero, one does have as an immediate consequence of Lemma 1 that $a'(t)\uparrow 0$ and $a''(t)\downarrow 0$ as $t\to\infty$.

LEMMA 2. *If* (1.2) *is satisfied and if $a(t)\not\equiv a(0)$, then either $-a'(t)$, $a''(t)>0$ for $0<t<\infty$ or there exists a $t_0>0$ such that $-a'(t)$, $a''(t)>0$ for $0<t<t_0$ and $a(t)\equiv a(t_0)=a(\infty)\geqq 0$ for $t_0\leqq t<\infty$.*

PROOF. If there exists a $t_0\geqq 0$ such that $a'(t_0)=0$, then as $-a'(t)$,

$a''(t) \geqq 0$ it follows that $a(t) \equiv a(t_0) \geqq 0$ for $t_0 \leqq t < \infty$. As $a(t) \not\equiv a(0)$, one must have $t_0 > 0$.

Hence, there exists a $t_1 > 0$ such that $a'(t_1) < 0$ and thus $a'(t) \leqq a'(t_1) < 0$ for $0 < t \leqq t_1$. Suppose $a''(t_1) = 0$. Then as $-a'''(t)$, $a''(t) \geqq 0$ it follows that $a''(t) \equiv 0$ for $t_1 \leqq t < \infty$. Hence $a(t) = a(t_1) + a'(t_1)(t - t_1)$ for $t_1 \leqq t < \infty$, which contradicts $a(t) \geqq 0$ for t sufficiently large. Hence $a''(t_1) > 0$ and thus $a''(t) \geqq a''(t_1) > 0$ for $0 < t \leqq t_1$. The conclusion follows readily.

LEMMA 3. *Let* (1.2) *be satisfied. Then*

$$ta'(t) \to 0, \qquad t^2 a''(t) \to 0 \quad \textit{as} \quad t \to 0+, \tag{2.1}$$

and

$$a'(t), \qquad ta''(t), \qquad t^2 a'''(t) \in L_1(0, \infty). \tag{2.2}$$

PROOF. By the mean value theorem and the monotonicity of $a'(t)$ one has $a(t) - a(0) = ta'(\xi) \leqq ta'(t) \leqq 0$ $(0 < \xi < t < \infty)$, from which (2.1a) follows. By second differences and the monotonicity of $a''(t)$ one has $0 \leqq (t/2)^2 a''(t) \leqq (t/2)^2 a''(\xi) = a(t) - 2a(t/2) + a(0)$ $(0 < \xi < t < \infty)$, which yields (2.1b).

As the integrand is of constant sign, (2.2a) follows from

$$\int_0^\infty a'(\xi)d\xi = \lim_{t\to\infty;\epsilon\to 0+} \int_\epsilon^t a'(\xi)d\xi = \lim_{t\to\infty;\epsilon\to 0+} \{a(t) - a(\epsilon)\}$$
$$= a(\infty) - a(0).$$

One similarly obtains (2.2b) and (2.2c).

LEMMA 4. *Let $b(t)$ be defined on $0 < t \leqq T$, $b'(t)$ exist and be finite on $0 < t \leqq T$, and $b'(t) \in L_1(\epsilon, T)$ for each $0 < \epsilon < T$. Let $q(t, \tau)$, $q_t(t, \tau) \in C[0 \leqq \tau, t \leqq T]$ in the pair t, τ. Let $b(\epsilon)q(t+\epsilon, t) \to \gamma(t)$ as $\epsilon \to 0+$ on $0 < t < T$, where $\gamma(t) \in C[0, T]$. Let there exist $\phi(\xi) \in L_1(0, T)$ such that*

$$|b(\xi)q(t, t-\xi)|, \; |b'(\xi)q(t, t-\xi)|, \; |b(\xi)q_t(t, t-\xi)| \leqq \phi(\xi) \qquad (0 < \xi \leqq t \leqq T).$$

Then

$$f(t) = \int_0^t b(t-\tau)q(t, \tau)d\tau \in C'[0, T]$$

and $f'(t) = h(t)$, where

$$h(t) = \gamma(t) + \int_0^t b'(t-\tau)q(t, \tau)d\tau + \int_0^t b(t-\tau)\frac{\partial q}{\partial t}(t, \tau)d\tau \qquad (0 \leqq t \leqq T). \tag{2.3}$$

PROOF. Define $h(t)$ by (2.3). It follows readily from the hypothesis that $h(t)\in C[0, T]$. One has

$$\int_0^t h(s)ds = \int_0^t \gamma(s)ds + \int_0^t \left\{\int_\tau^t b'(s-\tau)q(s,\tau)ds\right\} d\tau + \int_0^t \left\{\int_0^s b(s-\tau)\frac{\partial q}{\partial s}(s,\tau)d\tau\right\} ds, \tag{2.4}$$

where the interchange of order of integration is easily justified by the hypothesis and Fubini's theorem. We note that the hypothesis implies that $b(t)$ is absolutely continuous for $0<\epsilon\leqq t\leqq T$; see [3, p. 368]. This yields the second equality in

$$\begin{aligned}\int_\tau^t & b'(s-\tau)q(s,\tau)ds \\ &= \lim_{\epsilon\to 0+}\int_{\tau+\epsilon}^t b'(s-\tau)q(s,\tau)ds \\ &= \lim_{\epsilon\to 0+}\left\{b(s-\tau)q(s,\tau)\Big|_{\tau+\epsilon}^t - \int_{\tau+\epsilon}^t b(s-\tau)\frac{\partial q}{\partial s}(s,\tau)ds\right\} \\ &= b(t-\tau)q(t,\tau) - \gamma(\tau) - \int_\tau^t b(s-\tau)\frac{\partial q}{\partial s}(s,\tau)ds \quad (0<\tau<t).\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), one obtains $\int_0^t h(s)ds = f(t)$ after an interchange of order of integration. The result now follows.

3. **Proof of Theorem 1.** When we refer to (1.1) and (1.5) in this proof, we mean the identities that result from substituting $u(t)$ into them. The possibility of none of $a'(0)$, $a''(0)$, $a'''(0)$ being finite necessitates that a little care be used in handling certain integrals that arise, as already noticed in §1. In all cases it will be evident that Lemmas 3, 4, and arguments of the type (2.5) supply the rigor. We will, therefore, tacitly assume such considerations whenever they are relevant.

Define

$$E(t) = G(u(t)) + \frac{1}{2}a(t)\left[\int_0^t g(u(\tau))d\tau\right]^2 - \frac{1}{2}\int_0^t a'(t-\tau)\left[\int_\tau^t g(u(s))ds\right]^2 d\tau \geqq 0. \tag{3.1}$$

A calculation involving (1.1) and an integration by parts yields

$$
\begin{aligned}
E'(t) = {} & \frac{1}{2}\, a'(t)\left[\int_0^t g(u(\tau))d\tau\right]^2 \\
& - \frac{1}{2}\int_0^t a''(t-\tau)\left[\int_\tau^t g(u(s))ds\right]^2 d\tau \leqq 0,
\end{aligned} \tag{3.2}
$$

which implies

$$G(u(t)) \leqq E(t) \leqq E(0) = G(u_0),$$

where $u_0 = u(0)$. From (1.3c) it follows that

$$|u(t)| \leqq K < \infty \qquad (0 \leqq t < \infty), \tag{3.3}$$

where $K = K(u_0)$. In succeeding formulas K will not necessarily be the same as in (3.3); however, as in (3.3), it will depend only on u_0 (and will tend to zero as $u_0 \to 0$).

From (1.5), (2.2a), and (3.3) one observes that

$$|u''(t)| \leqq K < \infty \qquad (0 \leqq t < \infty). \tag{3.4}$$

Inequalities (3.3), (3.4) and the mean value theorem yield

$$|u'(t)| \leqq K < \infty \qquad (0 \leqq t < \infty). \tag{3.5}$$

A calculation involving (1.5) and an integration by parts yields

$$
\begin{aligned}
E''(t) = {} & \frac{1}{2}\, a''(t)\left[\int_0^t g(u(\tau))d\tau\right]^2 \\
& - \frac{1}{2}\int_0^t a'''(t-\tau)\left[\int_\tau^t g(u(s))ds\right]^2 d\tau \\
& - g(u(t))[u''(t) + a(0)g(u(t))].
\end{aligned}
$$

From the proof of Lemma 3 one observes that $t^2a''(t)$ is bounded on $0<t<\infty$. This, together with (3.3), (3.4), and Lemma 3 implies $|E''(t)| \leqq K < \infty$ $(0<t<\infty)$. Lemma 1, (3.1), and (3.2) now imply $E'(t)\to 0$ as $t\to\infty$. Hence

$$\lim_{t\to\infty} \int_0^t a''(t-\tau)\left[\int_\tau^t g(u(s))ds\right]^2 d\tau = 0,$$

which as $-a'''(t)$, $a''(t) \geqq 0$ yields

$$\lim_{t\to\infty} a''(T)\int_{t-T}^t \left[\int_\tau^t g(u(s))ds\right]^2 d\tau = 0$$

for every $0<T<\infty$. Choose $T_0>0$ arbitrarily if the first alternative of Lemma 2 holds and choose $0<T_0<t_0$ if the second one does. Then clearly

$$\lim_{t\to\infty}\int_{t-T}^{t}\left[\int_{\tau}^{t}g(u(s))ds\right]^{2}d\tau=0\qquad(0\leqq T\leqq T_0). \tag{3.6}$$

Suppose $u(t)\not\to 0$ as $t\to\infty$. Then there exist a $\lambda>0$ and a sequence $\{t_n\}$, where $t_1>0$ and $t_n\to\infty$ as $n\to\infty$, such that $|u(t_n)|\geqq\lambda$. Together with (1.3), (3.5), and the mean value theorem this implies the existence of a $\delta>0$ and a $\mu>0$, where $0<\delta\leqq\min(T_0, t_1)$, such that $|g(u(s))|\geqq\mu$ for $t_n-\delta\leqq s\leqq t_n$. Hence

$$\int_{t_n-\delta}^{t_n}\left[\int_{\tau}^{t_n}g(u(s))ds\right]^{2}d\tau\geqq\mu^2\int_{t_n-\delta}^{t_n}(t_n-\tau)^2d\tau=\frac{\mu^2\delta^3}{3}>0$$

$$(n=1,2,\cdots),$$

which contradicts (3.6). Thus, (1.4) for $j=0$ is established.

Formula (1.4), $j=1$, follows from (1.4), $j=0$, (3.4), and the mean value theorem (by reasoning similar to the proof of Lemma 1). Formula (1.4), $j=2$, follows from (1.4), $j=0$, (1.3), (1.5), and (2.2a).

4. **Proof of Theorem 2.** (i) We show first that $w(t)$ is bounded on $0\leqq t<\infty$. Suppose the contrary. Then there exists a sequence $\{t_n\}$, where $t_n\to\infty$ as $n\to\infty$, such that $|w(t_n)|\uparrow\infty$ as $n\to\infty$ and $|w(t)|\leqq|w(t_n)|$ for $0\leqq t\leqq t_n$. From (1.4), $j=0$, it follows that there exists a sequence $\{T_n\}$, where $T_n\leqq t_n$ and $T_n\to\infty$ as $n\to\infty$, such that $|w(t)|\geqq\frac{1}{2}|w(t_n)|$ for $t_n-T_n\leqq t\leqq t_n$. From (1.7) one has

$$1-u(t_n)=\int_0^{t_n-T_n}a(t_n-\tau)w(\tau)d\tau+\int_{t_n-T_n}^{t_n}a(t_n-\tau)w(\tau)d\tau. \tag{4.1}$$

Now

$$\left|\int_0^{t_n-T_n}a(t_n-\tau)w(\tau)d\tau\right|\leqq|w(t_n)|\int_{T_n}^{t_n}a(\tau)d\tau\leqq|w(t_n)|\int_{T_n}^{\infty}a(\tau)d\tau,$$

$$\left|\int_{t_n-T_n}^{t_n}a(t_n-\tau)w(\tau)d\tau\right|\geqq\frac{1}{2}|w(t_n)|\int_0^{T_n}a(\tau)d\tau,$$

and hence

$$|1-u(t_n)|\geqq\frac{1}{2}|w(t_n)|\left\{\int_0^{T_n}a(\tau)d\tau-2\int_{T_n}^{\infty}a(\tau)d\tau\right\}.$$

The last inequality clearly contradicts $u(t_n)\to 0$, $|w(t_n)|\to\infty$ as $n\to\infty$. Thus $w(t)$ is bounded on $0\leqq t<\infty$.

Let $\{t_n\}$ be any sequence such that $t_n \to \infty$ as $n \to \infty$. From (1.4), $j=0$, it follows that there exists a sequence $\{T_n\}$, where $T_n \leqq t_n$ and $T_n \to \infty$ as $n \to \infty$, such that $\max_{t_n - T_n \leq t \leq t_n} |w(t) - w(t_n)| \to 0$ as $n \to \infty$. Similar to (4.1) one has

$$1 - u(t_n) = \int_0^{t_n - T_n} a(t_n - \tau)w(\tau)d\tau + \int_{t_n - T_n}^{t_n} a(t_n - \tau)\{w(\tau) - w(t_n)\}d\tau$$
$$+ w(t_n)\int_{t_n - T_n}^{t_n} a(t_n - \tau)d\tau.$$

It now follows readily that $w(t_n) \to \{\int_0^\infty a(\tau)d\tau\}^{-1}$ as $n \to \infty$, which establishes (1.8).

(ii) Let $\rho(t) = \int_0^t w(\tau)d\tau$. Define

$$E_\rho(t) = \frac{1}{2}(\rho'(t))^2 + \frac{1}{2}a(t)\rho^2(t) - \frac{1}{2}\int_0^t a'(t - \tau)\{\rho(\tau) - \rho(t)\}^2 d\tau.$$

A straightforward calculation yields

$$E_\rho'(t) = -\frac{1}{2}\int_0^t a''(t - \tau)\{\rho(\tau) - \rho(t)\}^2 d\tau + \frac{1}{2}a'(t)\rho^2(t) + \rho'(t) \leqq \rho'(t),$$

from which it follows that

$$0 \leqq \frac{1}{2}a(\infty)\rho^2(t) \leqq \frac{1}{2}a(t)\rho^2(t) \leqq E_\rho(t) \leqq \rho(t).$$

Hence, $0 \leqq \rho(t) \leqq 2[a(\infty)]^{-1}$ for $0 \leqq t < \infty$.

Suppose $w(t) \not\to 0$ as $t \to \infty$. Then there exist a $\lambda > 0$ and a sequence $\{t_n\}$, where $t_n \to \infty$ as $n \to \infty$, such that $|w(t_n)| \geqq \lambda$. From (1.4), $j=0$, it follows that there exists a sequence $\{T_n\}$, where $T_n \to \infty$ as $n \to \infty$, such that $|w(t)| \geqq \lambda/2$ for $t_n \leqq t \leqq t_n + T_n$. This is easily seen (from the definition of $\rho(t)$) to contradict the boundedness of $\rho(t)$. Thus, (1.9) is established.

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