

## CONCERNING REAL NUMBERS WHOSE POWERS HAVE NONINTEGRAL DIFFERENCES

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Various classes of real numbers  $\alpha$  have been considered for which  $\alpha^s - \alpha^r$  ( $s, r$  positive integers,  $s \neq r$ ) is never integral (see pp. 244-246 of [1]). This note adds another class to this category.

If we consider the numbers  $\alpha = a + b(2)^{1/2}$ , we find that if  $a$  and  $b$  are positive and rational,  $a \geq 1$ , then  $\alpha^s - \alpha^r$  is nonintegral. This result is now generalized:

**THEOREM.** *Let  $\beta$  be any algebraic number having a minimal polynomial of the form:*

$$(1) \quad x^n - b_1x^{n-1} - b_2x^{n-2} - \dots - b_{n-1}x - b_n,$$

where each  $b_i$  is rational,  $b_i \geq 0$ ,  $n > 1$ . Let  $\alpha$  be defined by

$$(2) \quad \alpha = c_0 + c_1\beta^{d_1} + c_2\beta^{d_2} + \dots + c_m\beta^{d_m},$$

where each  $c_i > 0$ ,  $c_0 \geq 1$ ,  $m \geq 1$ ,  $1 \leq d_1 < d_2 < \dots < d_m < n$ ;  $c_i$  rational,  $d_i$  integral.

Then  $\alpha^s - \alpha^r$  is nonintegral for all positive integers  $s, r$  ( $s \neq r$ ).

**PROOF.** Let us suppose that

$$(3) \quad \alpha^s - \alpha^r - k = 0,$$

with  $s > r$  and  $k$  integral. Now,

$$(4) \quad \alpha^s = \sum_{\sum_{i=1}^m q_i = s} \frac{s!}{q_0!q_1! \dots q_m!} (c_0)^{q_0} (c_1\beta^{d_1})^{q_1} (c_2\beta^{d_2})^{q_2} \dots (c_m\beta^{d_m})^{q_m},$$

while

$$(5) \quad \alpha^r = \sum_{\sum_{i=1}^m t_i = r} \frac{r!}{t_0!t_1! \dots t_m!} (c_0)^{t_0} (c_1\beta^{d_1})^{t_1} (c_2\beta^{d_2})^{t_2} \dots (c_m\beta^{d_m})^{t_m}.$$

The right member of (5) contains the term for which  $t_0 = r - 1$ ,  $t_1 = 1$ , and all other  $t_i = 0$ , namely,

$$(6) \quad r(c_0)^{r-1}c_1\beta^{d_1};$$

similarly, the right member of (4) contains the term

$$(7) \quad s(c_0)^{s-1}c_1\beta^{d_1}.$$

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We note that the coefficient of  $\beta^{d_1}$  in (7) exceeds that in (6).

In the same way, suppose we select any particular term

$$(8) \quad \frac{r!}{t_0!t_1! \cdots t_m!} (c_0)^{t_0}(c_1\beta^{d_1})^{t_1}(c_2\beta^{d_2})^{t_2} \cdots (c_m\beta^{d_m})^{t_m}$$

in the right member of (5), where  $\sum_0^m t_i = r$ . Now, let us associate with (8) the term

$$(9) \quad \frac{s!}{q_0!q_1! \cdots q_m!} (c_0)^{q_0}(c_1\beta^{d_1})^{q_1}(c_2\beta^{d_2})^{q_2} \cdots (c_m\beta^{d_m})^{q_m}$$

of (4), where we pick  $q_0 = t_0 + s - r$ ,  $q_i = t_i$  for  $1 \leq i \leq m$ , so that  $\sum_0^m q_i = s$ .

We note that (8) and (9) both contain the same total exponent of  $\beta$ , namely  $\sum_1^m d_i t_i$ , but (except for the case  $c_0 = 1, t_1 = t_2 = \cdots = t_m = 0$ ) the coefficient in (9) is greater than that in (8). Also, any two terms of (5) have *distinct* corresponding terms in (4). Consequently, substituting (4) and (5) into (3), we obtain an equation in  $\beta$  of the form

$$(10) \quad g_0\beta^{d_m s} + g_1\beta^{d_m s - 1} + \cdots + g_{d_m s - d_1}\beta^{d_1} + \cdots + g_{d_m s - 1}\beta + g_{d_m s} = 0$$

where  $g_i \geq 0$  for  $i = 0, 1, 2, \dots, d_m s - 1$ , and  $g_{d_m s - d_1} > 0$ .

Now, by (1), each power of  $\beta$  in (10) can be expressed in terms of the basis  $\{1, \beta, \beta^2, \dots, \beta^{n-1}\}$  with *nonnegative* coefficients. After these substitutions have been made, the *total* coefficient of  $\beta^{d_1}$  will *still* be positive. Thus (10) becomes a polynomial in  $\beta$  whose degree  $d$  satisfies  $1 \leq d_1 \leq d < n$ , which is impossible.

REFERENCE

1. F. Supnick, H. J. Cohen and J. F. Keston, *On the powers of a real number reduced modulo one*, Trans. Amer. Math. Soc. **94** (1960), 244-257.

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