CONCERNING REAL NUMBERS WHOSE POWERS HAVE NONINTEGRAL DIFFERENCES

HERMAN J. COHEN AND FRED SUPNICK

Various classes of real numbers α have been considered for which $\alpha^s - \alpha^r$ (s, r positive integers, $s \neq r$) is never integral (see pp. 244–246 of [1]). This note adds another class to this category.

If we consider the numbers $\alpha = a + b(2)^{1/2}$, we find that if a and b are positive and rational, $a \geqq 1$, then $\alpha^s - \alpha^r$ is nonintegral. This result is now generalized:

THEOREM. *Let β be any algebraic number having a minimal polynomial of the form:*

$$x^n - b_1 x^{n-1} - b_2 x^{n-2} - \cdots - b_{n-1} x - b_n, \tag{1}$$

where each b_i is rational, $b_i \geqq 0$, $n > 1$. Let α be defined by

$$\alpha = c_0 + c_1 \beta^{d_1} + c_2 \beta^{d_2} + \cdots + c_m \beta^{d_m}, \tag{2}$$

where each $c_i > 0$, $c_0 \geqq 1$, $m \geqq 1$, $1 \leqq d_1 < d_2 < \cdots < d_m < n$; c_i rational, d_i integral.

Then $\alpha^s - \alpha^r$ is nonintegral for all positive integers s, r ($s \neq r$).

PROOF. Let us suppose that

$$\alpha^s - \alpha^r - k = 0, \tag{3}$$

with $s > r$ and k integral. Now,

$$\alpha^s = \sum_{\Sigma_0^m q_i = s} \frac{s!}{q_0! q_1! \cdots q_m!} (c_0)^{q_0} (c_1 \beta^{d_1})^{q_1} (c_2 \beta^{d_2})^{q_2} \cdots (c_m \beta^{d_m})^{q_m}, \tag{4}$$

while

$$\alpha^r = \sum_{\Sigma_0^m t_i = r} \frac{r!}{t_0! t_1! \cdots t_m!} (c_0)^{t_0} (c_1 \beta^{d_1})^{t_1} (c_2 \beta^{d_2})^{t_2} \cdots (c_m \beta^{d_m})^{t_m}. \tag{5}$$

The right member of (5) contains the term for which $t_0 = r - 1$, $t_1 = 1$, and all other $t_i = 0$, namely,

$$r (c_0)^{r-1} c_1 \beta^{d_1}; \tag{6}$$

similarly, the right member of (4) contains the term

$$s (c_0)^{s-1} c_1 \beta^{d_1}. \tag{7}$$

Presented to the Society, August 29, 1962; received by the editors June 1, 1962.

We note that the coefficient of β^{d_1} in (7) *exceeds* that in (6).

In the same way, suppose we select any particular term

$$\text{(8)} \qquad \frac{r!}{t_0!t_1!\cdots t_m!}(c_0)^{t_0}(c_1\beta^{d_1})^{t_1}(c_2\beta^{d_2})^{t_2}\cdots(c_m\beta^{d_m})^{t_m}$$

in the right member of (5), where $\sum_0^m t_i=r$. Now, let us associate with (8) the term

$$\text{(9)} \qquad \frac{s!}{q_0!q_1!\cdots q_m!}(c_0)^{q_0}(c_1\beta^{d_1})^{q_1}(c_2\beta^{d_2})^{q_2}\cdots(c_m\beta^{d_m})^{q_m}$$

of (4), where we pick $q_0 = t_0 + s - r$, $q_i = t_i$ for $1 \leqq i \leqq m$, so that $\sum_0^m q_i=s$.

We note that (8) and (9) both contain the same total exponent of β, namely $\sum_1^m d_it_i$, but (except for the case $c_0=1$, $t_1=t_2=\cdots=t_m=0$) the coefficient in (9) is greater than that in (8). Also, any two terms of (5) have *distinct* corresponding terms in (4). Consequently, substituting (4) and (5) into (3), we obtain an equation in β of the form

$$\text{(10)} \qquad g_0\beta^{d_ms}+g_1\beta^{d_ms-1}+\cdots+g_{d_ms-d_1}\beta^{d_1}+\cdots+g_{d_ms-1}\beta+g_{d_ms}=0$$

where $g_i\geqq 0$ for $i=0, 1, 2, \cdots, d_ms-1$, and $g_{d_ms-d_1}>0$.

Now, by (1), each power of β in (10) can be expressed in terms of the basis $\{1, \beta, \beta^2, \cdots, \beta^{n-1}\}$ with *nonnegative* coefficients. After these substitutions have been made, the *total* coefficient of β^{d_1} will *still* be positive. Thus (10) becomes a polynomial in β whose degree d satisfies $1\leqq d_1\leqq d<n$, which is impossible.

Reference

1. F. Supnick, H. J. Cohen and J. F. Keston, *On the powers of a real number reduced modulo one*, Trans. Amer. Math. Soc. **94** (1960), 244–257.

City College