A NOTE ON THE IDEAL \([uv]\)

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If \(P \in [uv]\) has signature\(^1\) \((d_1, d_2)\) and weight \(d_1d_2\), by Levi’s reduction process \([1, \text{p. 561}]\) it is known that \(P = c_1u^{d_1}v^{d_2}\) for some number \(c_1\).\(^2\) If one were to use the symmetric reduction process (obtained by interchanging the roles of \(u\) and \(v\)) he would find \(P = c_2u^{d_2}v^{d_1}\); hence \(u^{d_1}v^{d_2} = m u^{d_2}v^{d_1}\) for some number \(m\). In \([3]\) the number \(m\) is obtained when \(d_1 = d_2\). The number \(m\) for all positive \(d_1\) and \(d_2\) is given by the

**Theorem.** \(u^{d_1}v^{d_2} = (-1)^{d_1d_2}[(d_1!)/d_2!][(d_2!)/d_1!]u^{d_2}v^{d_1}\).

**Lemma 1(a).** If \(P\) involves both \(u\) and \(v\) and \(P_{ui}\) and \(P_{vj}\) have excess weights \([2, \text{p. 426}]\) of 0 and 1 respectively, then:

\[ P_{ui}v_j = (-j/(i + 1))P_{ui+1}v_{j-1}. \]

**Proof.**

\[
P_{ui}v_j = -P\left[\sum_{k=1}^{i} \frac{ilj!}{(i - k)!(j + k)!}u^{i-k}v_{j+k} + \frac{ilj!}{(i + 1)!(j - 1)!}u^{i+1}v_{j-1}\right].
\]

The typical term in the first sum contains \(P_{ui-k}, \, k \geq 1\), which has negative excess weight and thus is in \([uv]\). Similarly, the typical term in the second sum contains \(P_{v_{j-k}, \, k \geq 2}\), and, having negative excess weight, is in \([uv]\).

**Lemma 1(b).** If \(Q\) is a pp. in \(v\) alone and \(Q_{ujv_1}\) is of excess weight 0, then

\[ Q_{ujv_1} = -\frac{1}{j + 1} Q_{uj+1v}. \]

The proof is so similar to the one just given that the details will be omitted.

Assume \(d_1\) and \(d_2\) are fixed positive integers. For \(0 \leq i \leq d_2, \, 0 \leq s \leq d_1\) and for \(i = d_2, \, s = 0\) let

\[
R(i, s) = u_i^{d_1-s} u_{i+1}^{s} v_d^{d_2-i} v_{d_1-i}^{d_2-(i+1)}.\]

\(^1\) Some familiarity with the nomenclature employed in \([1]\) and \([2]\) is assumed.

\(^2\) In this note all congruences are modulo \([uv]\).
Lemma 2.

\[ u_{d_1} v_{d_2} \equiv (-1)^{d_1+s} \frac{(d_1!)^{i+1}}{(i!)^{d_1-s}(i+1)^s(d_1-s)!} R(i, s). \]

Proof. The lemma is clearly true if \( i = s = 0 \), and the proof is completed using induction on the pair \((i, s)\) ordered lexicographically. We assume the lemma valid for the pair \((i, s)\) and distinguish two cases.

First, if \( 0 \leq s < d_1 - 1 \), we need only show

\[ R(i, s) = - \frac{d_1 - s}{i + 1} R(i, s + 1). \]

By Lemma 1(a) with \( P = u^{d_1-s-1} v^i \) and \( j = d_1 - s \)

\[ R(i, s) = u_{d_1-s-1} v^i u_1 v_1 v_{d_1-s-1} \frac{d_1-(i+1)}{i+1} \]

\[ = - \frac{d_1 - s}{i + 1} u_i v_{d_1-s-1} u_{i+1} v_{d_1} \]

\[ = - \frac{d_1 - s}{i + 1} R(i, s + 1). \]

Second, if \( s = d_1 - 1 \), by Lemma 1(b) with \( Q = v^j \) we see

\[ R(j, d_1 - 1) = v^j u_{d_1-1} v_{d_1} = - \frac{1}{j + 1} u_{j+1} v_{d_1}. \]

Combined with the induction assumption, this completes the proof of Lemma 2. The validity of the theorem, which is a special case of Lemma 2 (when \( i = d_1 \), \( s = 0 \)), has thus been demonstrated.

Bibliography