A NOTE ON THE IDEAL $[uv]$

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If $P \in [uv]$ has signature[1] (d_1, d_2) and weight $d_1 d_2$, by Levi's reduction process [1, p. 561] it is known that $P \equiv c_1 u^{d_1} v_{d_1}^{d_2}$ for some number c_1.[2] If one were to use the symmetric reduction process (obtained by interchanging the roles of u and v) he would find $P \equiv c_2 u_{d_2}^{d_1} v^{d_2}$; hence $u^{d_1} v_{d_1}^{d_2} \equiv m u_{d_2}^{d_1} v^{d_2}$ for some number m. In [3] the number m is obtained when $d_1 = d_2$. The number m for all positive d_1 and d_2 is given by the

THEOREM. $u^{d_1} v_{d_1}^{d_2} \equiv (-1)^{d_1 d_2} [(d_1!)^{d_2}/(d_2!)^{d_1}] u_{d_2}^{d_1} v^{d_2}$.

LEMMA I(a). *If P involves both u and v and Pu_i and Pv_j have excess weights* [2, p. 426] *of* 0 *and* 1 *respectively, then*:

$$Pu_i v_j \equiv (-j/(i+1)) P u_{i+1} v_{j-1}.$$

PROOF.

$$Pu_i v_j \equiv -P\left[\left(\sum_{k=1}^{i} \frac{i!j!}{(i-k)!(j+k)!} u_{i-k} v_{j+k}\right) + \frac{i!j!}{(i+1)!(j-1)!} u_{i+1} v_{j-1} + \sum_{k=2}^{j} \frac{i!j!}{(i+k)!(j-k)!} u_{i+k} v_{j-k}\right].$$

The typical term in the first sum contains Pu_{i-k}, $k \geqq 1$, which has negative excess weight and thus is in $[uv]$. Similarly, the typical term in the second sum contains Pv_{j-k}, $k \geqq 2$, and, having negative excess weight, is in $[uv]$.

LEMMA I(b). *If Q is a pp. in v alone and $Qu_j v_1$ is of excess weight* 0, *then*

$$Qu_j v_1 \equiv -\frac{1}{j+1} Q u_{j+1} v.$$

The proof is so similar to the one just given that the details will be omitted.

Assume d_1 and d_2 are fixed positive integers. For $0 \leqq i < d_2$, $0 \leqq s < d_1$ and for $i = d_2$, $s = 0$ let

$$R(i, s) = u_i^{d_1 - s} u_{i+1}^{s} v^{i} v_{d_1 - s} v_{d_1}^{d_2 - (i+1)}.$$

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[1] Some familiarity with the nomenclature employed in [1] and [2] is assumed.

[2] In this note all congruences are modulo $[uv]$.

LEMMA 2.

$$u^{d_1} v_{d_1}^{d_2} \equiv (-1)^{id_1+s} \frac{(d_1!)^{i+1}}{(i!)^{d_1-s}(i+1)^s(d_1-s)!} R(i, s).$$

PROOF. The lemma is clearly true if $i=s=0$, and the proof is completed using induction on the pair (i, s) ordered lexicographically. We assume the lemma valid for the pair (i, s) and distinguish two cases.

First, if $0 \leqq s < d_1 - 1$, we need only show

$$R(i, s) \equiv -\frac{d_1 - s}{i+1} R(i, s+1).$$

By Lemma I(a) with $P = u_i^{d_1-s-1} v^i$ and $j = d_1 - s$

$$\begin{aligned} R(i, s) &= u_i^{d_1-s-1} v^i u_i v_{d_1-s} u_{i+1}^{s} v^{d_2-(i+1)} \\ &\equiv -\frac{d_1 - s}{i+1} u_i^{d_1-s-1} v^i u_{i+1} v_{d_1-s-1} u_{i+1}^{s} v_{d_1}^{d_2-(i+1)} \\ &= -\frac{d_1 - s}{i+1} R(i, s+1). \end{aligned}$$

Second, if $s = d_1 - 1$, by Lemma I(b) with $Q = v^j$ we see

$$R(j, d_1 - 1) = v^j u_j v_1 u_{j+1}^{d_1-1} v_{d_1}^{d_2-(j+1)} \equiv -\frac{1}{j+1} u_{j+1}^{d_1} v^{j+1} v_{d_1}^{d_2-(j+1)}.$$

Combined with the induction assumption, this completes the proof of Lemma 2. The validity of the theorem, which is a special case of Lemma 2 (when $i = d_2$, $s = 0$), has thus been demonstrated.

BIBLIOGRAPHY

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