Recall that, by Theorem 3.1, $E(\mathfrak{Q})\supset\Delta(A)$. From our definitions $\Delta(A)$ is dense in $\mathfrak{Z}$ so that also $E(\mathfrak{Q})\supset\mathfrak{Z}$. Since the reverse inequality is clear, the proof is complete.

References

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SEMI-HOMOGENEOUS FUNCTIONS

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1. **Introduction and statement of results.** A function f is called *homogeneous of degree n with respect to the set A*, or briefly *semi-homogeneous* if

$$f(ax) = a^n f(x) \tag{1.0}$$

is satisfied for all x in the domain of f and all a in A.

With each admissible A there is associated a class of solutions of (1.0). E.g., let R denote the set of all real numbers and let f be a function on R to R. If A consists only of the irrationals, then $f(x) = cx^n$ $(c=f(1))$ is the unique solution of (1.0). On the other hand, if A consists only of the rationals, then in addition to $f(x)=cx^n$, (1.0) has other solutions (e.g., if n is any nonzero integer and $f(x)=x^n$ or 0 accordingly as x is rational or irrational).

We are interested in studying decompositions of the set of admissible A's into classes and in characterizing the solutions of (1.0) corresponding to these classes. In this paper we show how this can be done in a natural way for semi-homogeneous functions of a real variable. We note that in this case our methods apply to

$$f(ax) = p(a)f(x) \qquad (a \in A \subset R), \tag{1.1}$$

where p is a product-preserving function on R to R (cf. [1]). We shall therefore confine our attention to (1.1).

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Let $C(p, A)$ denote the set of all functions on R to R satisfying (1.1). Let $R^* = R - \{0\}$ and A^* be the multiplicative subgroup of R^* generated by $A - \{0\}$. It is easy to see that if A contains a set of generators for R^*, then $f(x) = f(1)p(x)$. Thus, the problem of determining which subsets of R lead to the solution $f(x) = f(1)p(x)$ of (1.1) is equivalent to classifying those subsets of R which generate R^* (cf. §5). With respect to the general case we first establish the following lemma.

LEMMA. *If $p \neq 0$ and $p \neq 1$ on A^*, then $C(p, A) = C(p, A^*)$ (cf. §2).*

Thus we are led to consider the problem: *For an arbitrary subgroup A^* of R^*, how can the members of $C(p, A^*)$ be characterized?*

The general answer to the above question is given in Theorem 1. Let R^*/A^* denote the multiplicative quotient group of R^* modulo A^*, B a set of representatives for R^*/A^*, and p a product-preserving function on R to R such that $p \neq 0$ and $p \neq 1$ on A^* (cf. [2]).

THEOREM 1. (i) *$f \in C(p, A^*)$ if and only if*

$$(1.2) \qquad f(x) = \begin{cases} F(b)p(x) & x = ab,\ a \in A^*,\ b \in B, \\ 0 & x = 0, \end{cases}$$

where F is a function on B to R.

(ii) *If $f \in C(p, A^*)$, then f is uniquely determined when it is known on B (cf. §2).*

An explicit classification and description of the solutions of (1.1) depends on a consideration of the different possibilities for A^* and p. We note that A^* is either dense in the positive reals, dense in R, or is discrete and countable. Also, if p is continuous on R^* ($p \neq 0$), then p is of the form $p(x) = |x|^k$, or $p(x) \in \{x^k\ (0 < x),\ -|x|^k\ (x < 0)\}$ where k is a real constant.

THEOREM 2. *Let $A^* \subset R^*$, $p \neq 0$, 1 on A^*, and $f \in C(p, A^*)$. If p is continuous on R and A^* is dense in R, then f is of type*

T_1: $f(x) = cp(x)$, "c" a constant,

T_2: f is continuous only at $x = 0$, and the closure $\text{Cl}(G(f))$ of its (cartesian) graph $G(f)$ (in E_2) is a set of curves $\{y = cp(x)\}$ with $\{c\}$ bounded, or

T_3: f is totally discontinuous and $\text{Cl}(G(f))$ is a set of curves $\{y = cp(x)\}$ with $\{c\}$ unbounded. $\text{Cl}(G(f))$ may be the entire plane if f is of type T_3 (cf. §3).

REMARK. It is of interest to note that if f is a function which satisfies $f(x+y)=f(x)+f(y)$, then the classic description of f given by G. Hamil (namely, that f is of the form $f(x)=cx$, or is totally discontinuous and the graph of f is dense in the entire plane) is contained within our Theorem 2 (cf. [3]). This follows from the fact that $f(x+y)=f(x)+f(y)$ implies f is homogeneous of degree 1 with respect to the rationals, i.e., $f(ax)=af(x)$ for all rational numbers a.

Let A^* be an infinite discrete subgroup of R^*. Let g denote the smallest member of A^* such that $1<g$, $G_0\equiv\{(x, f(x)): 1\leqq x<g\}$ where f is a function on R to R, D_a the transformation of E_2 into E_2 defined by $D_a(x, y)=(ax, p(a)y)$ $(a\in R)$, and D_aG_0 the image of G_0 under D_a.

THEOREM 3. *If A^* is an infinite discrete subgroup of R^* containing at least one negative member and $f\in C(p, A^*)$, then $\{D_aG_0: a\in A^*\cup\{0\}\}$ is a decomposition of $G(f)$ (cf. [4] and §4).*

In Theorems 2 and 3 we have considered the cases (i) p continuous and A^* dense, and (ii) p arbitrary and A^* discrete. We note that if p is discontinuous on R^*, then p is totally discontinuous and the graph of p is dense in the first and second or the first and third quadrants depending on whether $p(-1)=1$ or -1, respectively. The problem of obtaining an esthetic classification of the solutions of (1.1) for the case p discontinuous and A^* dense in R remains open.

2. Proof of the Lemma and Theorem 1.

PROOF OF THE LEMMA. Since $A-\{0\}\subset A^*$, $C(p, A^*)\subset C(p, A-\{0\})$. If $0\notin A$, then $C(p, A^*)\subset C(p, A)$. If $0\in A$, then since $p(0)=0$ every member of $C(p, A)$ must assign 0 to 0. But $p\not\equiv 1$ on A^* implies there exist $a, a'\in A^*$ such that $p(a)\neq p(a')$. Thus, $f(0)(p(a)-p(a'))=0$ yields $f(0)=0$ whenever $f\in C(p, A^*)$. Therefore, $C(p, A^*)\subset C(p, A)$.

Let $f\in C(p, A)$. To prove $C(p, A)\subset C(p, A^*)$ we use the fact that every member of A^* is a finite product of members of $A-\{0\}$ or their reciprocals. Thus, if $s\in A^*$, then $s=at$ or $s=a^{-1}t$ with $a\in A-\{0\}$ and $t\in A^*$. If $s=at$, then

$$f(sx) = f(atx) = p(a)f(tx). \tag{2.1}$$

If $s=a^{-1}t$, then

$$f(sx) = p(a^{-1}a)f(a^{-1}tx) = p(a^{-1})f(tx). \tag{2.2}$$

Applying operations of the form (2.1) or (2.2) successively we obtain $f(sx)=p(s)f(x)$ $(s\in A^*)$. Thus, $C(p, A)=C(p, A^*)$.

PROOF OF THEOREM 1 (i). Since R^*/A^* defines a partition of R^*, each nonzero real number x has a unique representation in terms of A^* and B. Namely, $x=ab$ $(a\in A^*,\ b\in B)$. Thus (1.2) defines a function on R to R.

Let $f\in C(p,\ A^*)$. Since $p\not\equiv 0$, then $p(b)\neq 0$ $(b\in R^*)$. Thus $f(x) = p(x)(p(b))^{-1}f(b)$ whenever $x=ab$ $(a\in A^*,\ b\in B)$. Since $p\not\equiv 1$ on A^*, then $f(0)=0$. Consequently, f can be expressed in the form (1.2) with F the function on B to R defined by $F(b)=(p(b))^{-1}f(b)$.

Conversely, let f be defined by (1.2). If $s\in A^*$ and $x\in R^*$, then $x=ab$ and $sx=sab$ $(a\in A^*,\ sa\in A^*,\ b\in B)$. Thus,

$$f(sx) = F(b)p(sx) = p(s)F(b)p(x) = p(s)f(x).$$

If $x=0$, then $f(s0)=0=p(s)f(0)$. Therefore, $f\in C(p,\ A^*)$.

Proof of Theorem 1 (ii). If $f, f'\in C(p,\ A^*)$ and $f(b)=f'(b)$ $(b\in B)$, then

$$f(x) = p(a)f(b) = p(a)f'(b) = f'(x)$$

whenever $x=ab$ $(a\in A^*,\ b\in B)$, and $f(0)=0=f'(0)$.

3. **Proof of Theorem 2.** By Theorem 1 the image under f of each coset of R^*/A^* lies on a curve, $y=cp(x)$. Specifically, if $x\in A^*b$, then $(x, f(x))$ lies on $y=F(b)p(x)$. Since each coset of R^*/A^* is dense in R, the image I of each coset is dense in the curve containing I. Each $y=cp(x)$ is continuous, hence the closure of the graph of f is a set of curves $K\equiv\{y=cp(x)\}$ through the origin. If all the cosets of R^*/A^* are mapped onto the same curve, then f is a function of type T_1. If this is not the case, then f is a function of type T_2 or T_3 accordingly as the set of coefficients $C\equiv\{c\}$ appearing in K is bounded or unbounded. If K contains at least two curves, it is clear that f cannot be continuous at nonzero values of x. Suppose C is bounded. If $c_1=\text{lub } C$, $c_2=\text{glb } C$, and $c_0=\max\{|c_1|,\ |c_2|\}$, then $f(x)$ lies between $-c_0p(x)$ and $c_0p(x)$ for all x. If $\{x_i\}$ is any sequence of real numbers converging to 0, then $\lim_{x_i\to 0}\{c_0p(x_i)\}=0$ implies $\lim_{x_i\to 0}\{f(x_i)\}=0$. Since $f(0)=0$, we have f is continuous at $x=0$. Hence, f is of type T_2. If C is unbounded, then f is unbounded in every neighborhood of the origin. Since $x=0$ is the only possible point of continuity and $f(0)=0$, f is totally discontinuous and is thus of type T_3.

If the index of A^* in R^* is $2^{\aleph_0}$ and B is a set of representatives for R^*/A^*, then there exists a 1-1 function F on B to R. The function f defined by $f(x)=F(b)p(x)$ $(x=ab,\ a\in A^*,\ b\in B)$ and $f(0)=0$ is a type T_3 member of $C(p,\ A^*)$ and $G(f)$ is dense in $K\equiv\{y=F(b)p(x): b\in B\}$. But $K\equiv\{y=rp(x): r\in R\}$ is the entire plane with the exception of the points on the y-axis having nonzero ordinates. Hence $\text{Cl}(G(f))$ is the entire plane. This completes the proof of Theorem 2.

4. Proof of Theorem 3. Since A^* is discrete and not finite there exists a smallest member $g \in A^*$ such that $1 < g$ and $B = \{x \in R: 1 \leqq x < g\}$ is a set of representatives for R^*/A^*.

A^* induces the following decomposition of R^*. The set of positive reals is decomposed into intervals $g^n \leqq x < g^{n+1}$ where n ranges over the integers. A^* also contains a largest member h such that $h \leqq -1$. We now decompose the set of negative reals into the sets $hg^{n+1} < x \leqq hg^n$. In this way R^* is decomposed into intervals each of which is the image of the set B under multiplication by a suitable member of A^*. This decomposition of R^* induces a decomposition of $G(f)$ into the following sets:

$$G_n = \{(x, f(x)): g^n \leqq x < g^{n+1}\}, \quad G_n' = \{(x, f(x)): hg^{n+1} < x \leqq hg^n\},$$

and $\{(0, 0)\}$, where g and h are as described above and n ranges over the integers.

We note that G_0 is the graph of f in E_2 which corresponds to B. Since f is completely determined when we know its values on B, we seek a relation between the set G_0 and the other sets of this decomposition of $G(f)$. Applying D_{g^n} to G_0 we obtain

$$\begin{aligned} \{(g^n x, p(g^n)f(x)): 1 \leqq x < g\} &= \{(g^n x, f(g^n x)): 1 \leqq x < g\} \\ &= \{(y, f(y)): g^n \leqq y < g^{n+1}\}. \end{aligned}$$

Therefore, $D_{g^n}G_0 = G_n$. Similarly, $D_{hg^n}G_0 = G_n'$ and $D_0G_0 = \{(0, 0)\}$.

$G(f)$ thus decomposes into sets each of which is the image of the set G_0 under a transformation D_a. Therefore, $\{D_aG_0: a \in A^* \cup \{0\}\}$ is a decomposition of $G(f)$. This completes the proof of Theorem 3.

Theorems analogous to Theorems 2 and 3 may be obtained when A^* is a subgroup of the positive reals or when p is continuous only on R^*.

5. Remarks on the solution $f(1)p(x)$. Let $A \subset R$ and μ denote the Lebesgue measure on R.

1. *If $\mu(A) > 0$ and A contains at least one negative member, then $f(1)p(x)$ is the unique solution of* (1.1).

PROOF. Let R_p^* denote the positive reals. If $A_p = A^* \cap R_p^*$, then $\mu(A_p) > 0$. Since $\mu(\ln[A_p]) > 0$, we have

$$D(\ln[A_p]) \equiv \{x - y: x = \ln(a), y = \ln(b)\ a, b \in A_p\}$$

contains an open interval containing the origin (cf. [5]). Thus,

$$e^{D(\ln[A_p])} = \{e^{\ln(a) - \ln(b)} = a/b: a, b \in A_p\}$$

contains an open interval containing 1. Since every positive real number has an nth root in this interval, $e^{D(\ln[A_p])}$ generates R_p^*. Hence, $R_p^* \subset A^*$. Since A contains a negative number $-1 \in A^*$. Therefore, $A^* = R^*$ and the assertion follows from the Lemma.

2. *There exist sets* $A \subset R$ *such that* $\mu(A) = 0$ *and* $f(1)p(x)$ *is the unique solution of* (1.1)

PROOF. Let C denote the Cantor set and $A = \{e^c: c \in C\} \cup \{-1\}$. It is a theorem of Steinhaus that for every number d between 0 and 1 there are two numbers of C which differ by d (cf. [6]). Thus, $\{e^c: c \in C\}$ generates R_p^*. Hence, $A^* = R^*$.

3. It is clear that if A is countable then $A^* \neq R^*$. We also note that R^* does not contain any minimal set of generators.

NOTES AND REFERENCES

1. The special case $p(a) = a$ ($a \in A \subset R$) was first considered by the authors and the results announced in Abstract 577-8, Notices Amer. Math. Soc. **8** (1961), 51.

2. If $0 \in A$ and $p \equiv 1$, then $C(1, A)$ is the set of all constant functions. If $0 \notin A$, A is not null, and $p \equiv 1$ on A^*, then $C(1, A)$ is the set of all functions which are constant on the cosets of R^*/A^* and $f(0)$ is an arbitrary constant.

3. G. Hamil, *Eine Basis aller Zahlen und die unstetige Lösungen der Funktionalgleichung* $f(x+y) = f(x) + f(y)$, Math. Ann. **60** (1905), 459–462.

4. By a *decomposition* of a set X we mean a disjoint family of subsets of X whose union is X.

5. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1950, Theorem B, p. 68.

6. H. Steinhaus, *A new property of G. Cantor's set*, Wektor **7** (1917). (Polish) See also, J. F. Randolph, *Distances between points of the Cantor set*, Amer. Math. Monthly **47** (1940), 549.

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