ON CERTAIN SEQUENCE TO SEQUENCE TRANSFORMATIONS WHICH PRESERVE CONVERGENCE

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1. **Introduction.** The purpose of this paper is to consider the following question. For what subsets U of the set S of all complex number sequences is it true that any triangular matrix A ($A_{pq}=0$ if $q>p$) which is convergence preserving over U is convergence preserving? Several subsets of S having this property are given here. The methods of proof generally avoid the three well-known conditions of Silverman and Toeplitz, namely,

$$(1.1)\quad \begin{aligned} &(1)\quad \{A_{nk}\}_{n=1}^{\infty} \text{ converges,} \qquad k = 1, 2, 3, \cdots, \\ &(2)\quad \left\{\sum_{p=1}^{n} A_{np}\right\}_{n=1}^{\infty} \text{ converges,} \\ &(3)\quad \text{there exists } M \text{ such that } \sum_{p=1}^{n} |A_{np}| < M, \quad n = 1, 2, 3, \cdots, \end{aligned}$$

which are necessary and sufficient for A to be convergence preserving. The obvious method of attack when a given subset U of S is being considered would seem to be to try to prove that if A is convergence preserving over U, then A satisfies (1.1). This method has neither been useful in making conjectures nor in proving them, except in one case. The more fruitful approach is to utilize the observations listed in the next section.

2. **Preliminaries.** The following notation will be used:

S_c: set of all convergent complex sequences.

S_{ck}: set of all real sequences convergent to k.

$\{k\}$: the sequence $\{a_p\}$ such that $a_i=k$, $i=1, 2, 3, \cdots$.

T: set of all sequences $\{a_p\}$ such that either $\{a_p\}\in S_{c0}$ or $\{a_p\} =\{1\}$.

S_{ck+}: set of all real sequences $\{a_p\}$ convergent to k such that $a_i\geqq k$, $i=1, 2, 3, \cdots$.

$S_{ck\downarrow}$: set of all real sequences $\{a_p\}$ convergent to k such that $a_i\geqq a_{i+1}$, $i=1, 2, 3, \cdots$.

S_{BV}: set of all sequences $\{a_p\}$ such that $\sum |a_{p+1}-a_p|$ converges.

DEFINITION. The triangular matrix A is convergence preserving (abbreviated: A is c.p.) means if $x\in S_c$, then $Ax\in S_c$.

Presented to the Society, April 23, 1962; received by the editors April 28, 1962.

DEFINITION. If U is a subset of S which contains an element of S_c, then the statement that the triangular matrix A is convergence preserving over U (abbreviated: A is c.p.o. U) means if $x \in U \cdot S_c$, then $Ax \in S_c$.

DEFINITION. Suppose $\{a_p\}$ converges to a and $\{b_p\}$ converges to b. The statement that $\{b_p\}$ converges faster (slower) than $\{a_p\}$ means $\operatorname{Lim} (b_p - b)/(a_p - a) = 0$ ($\operatorname{Lim} (a_p - a)/(b_p - b) = 0$). (Compare with [**2**, p. 128].)

As indicated in the introduction, most of the proofs in the next section depend on the following trivially obvious observations.

OBSERVATION 2.1. The set of all finite linear combinations of elements of T with complex coefficients is S_c.

OBSERVATION 2.2. If A is c.p.o. T, then A is c.p.

PROOF. Follows from Observation 2.1 and the fact that A is a linear transformation defined over S.

EXAMPLE. A, with $A_{pq} = 0$ if $p \neq q$, $A_{pq} = (-1)^{p+1}$ if $p = q$, is c.p.o. S_{c0}, but is not c.p.

OBSERVATION 2.3. If A is c.p.o. the set of all sequences with positive terms, then A is c.p.

OBSERVATION 2.4. If A is c.p.o. S_{ck} and $k \neq 0$, then A is c.p.

OBSERVATION 2.5. If A is c.p.o. S_{c0+} and $A\{1\} \in S_c$, then A is c.p.

OBSERVATION 2.6. If A is c.p.o. S_{ck+} and $k \neq 0$, then A is c.p.

3. **Theorems.** We begin with a theorem which gives a condition that is generally not strong enough to make A convergence preserving.

THEOREM 3.1. *If A is c.p.o. $S_{ck\downarrow}$ and $k \neq 0$, then A is c.p.o. S_{BV}.*

PROOF. Suppose the real sequence $\{a_p\} \in S_{BV}$. Let $b_1 = |a_1|$ and $b_n = |a_1| + \sum_{p=1}^{n-1} |a_{p+1} - a_p|$, $n = 2, 3, 4, \cdots$, and let $c_n = b_n - a_n$, $n = 1, 2, 3, \cdots$. Then $\{b_p\}$ converges, say to b, and $\{c_p\}$ converges, say to c. Let $b_n' = b + k - b_n$ and $c_n' = c + k - c_n$, $n = 1, 2, 3, \cdots$. Then $\{b_p'\} \in S_{ck\downarrow}$, $\{c_p'\} \in S_{ck\downarrow}$, and $a_n = c_n' - b_n' + b - c$, $n = 1, 2, 3, \cdots$. Therefore, $A\{a_p\} \in S_c$, since $A\{b - c\} = [(b-c)/k]A\{k\}$. The complex case follows from the fact that if $\{z_p\} \in S_{BV}$ and $z_p = x_p + iy_p$, $p = 1, 2, 3, \cdots$, where each of $\{x_p\}$ and $\{y_p\}$ is a real sequence, then $\{x_p\} \in S_{BV}$ and $\{y_p\} \in S_{BV}$.

COROLLARY 3.1a. *If A is c.p.o. $S_{c0\downarrow}$ and $A\{1\} \in S_c$, then A is c.p.o. S_{BV}.*

EXAMPLE. Corollary 3.1a is the best possible in a certain sense as shown by the following example: Let $A_{pq} = 0$ if $q > p$, $A_{pq} = (-1)^{q+1}q^{-1}$

if $q \leqq p$. Clearly $A\{1\} \in S_c$. Suppose $\{x_p\} \in S_{c0\downarrow}$. The fact that $A\{x_p\} \in S_c$ follows from a theorem of Abel [2, p. 137]. Thus A satisfies the hypothesis of Corollary 3.1a and is therefore convergence preserving over S_{BV}. But A is not c.p.

THEOREM 3.2. *Suppose* X *is a real sequence which converges to* 1. *If* A *is convergence preserving over the set of all real sequences* $\{a_p\}$ *convergent to* 1 *such that* $|X_p - 1| \leqq |a_p - 1|$, $p = 1, 2, 3, \cdots$, *then* A *is c.p.*

PROOF. Suppose $\{a_p\} \in S_{c0}$. If p is a positive integer, let $b_p = X_p$ if $(X_p - 1)a_p \geqq 0$, $b_p = 2 - X_p$ if $(X_p - 1)a_p < 0$, and let $c_p = b_p + a_p$. Clearly $|X_p - 1| = |b_p - 1|$, $|X_p - 1| \leqq |c_p - 1|$, $p = 1, 2, 3, \cdots$, and $\{b_p\}$ and $\{c_p\}$ converge to 1. Therefore $A\{b_p\} \in S_c$ and $A\{c_p\} \in S_c$. Thus $A\{a_p\} \in S_c$, and therefore A is c.p.o. S_{c0}. But $\{X_p - 1\} \in S_{c0}$. Hence $A\{X_p - 1\} \in S_c$. Therefore $A\{1\} \in S_c$, since $A\{X_p\} \in S_c$. Hence by Observation 2.2, A is c.p.

We consider the next theorem mainly for comparison with Theorem 3.2.

THEOREM 3.3. *Suppose* X *is a real sequence which converges to* 1 *such that* $X_p = 1$ *for no* p. *If* A *is c.p.o. the set of all real sequences* $\{a_p\}$ *such that* $|a_p - 1| \leqq |X_p - 1|$, $p = 1, 2, 3, \cdots$, *then* A *satisfies* (1) *and* (2) *of* (1.1) *but is not necessarily c.p.*

PROOF. It follows from the hypothesis that $A\{1\} \in S_c$. Hence (2) of (1.1) holds. Thus it is clear that if $\{a_p\}$ is a real sequence such that $|(a_p + 1) - 1| \leqq |X_p - 1|$, $p = 1, 2, 3, \cdots$, then $A\{a_p\} \in S_c$. Suppose k is a positive integer. Let $a_p = 0$ if $p \neq k$ and $a_k = (1/2)|X_k - 1|$. Then $A\{a_p\} \in S_c$. Hence $\{A_{nk}\}_{n=1}^{\infty}$ converges, and so (1) of (1.1) holds. To see that A is not necessarily convergence preserving, consider the matrix A defined in the example following Corollary 3.1a. Let $X_p = 1 + 1/p$, $p = 1, 2, 3, \cdots$. Since $A\{1\} \in S_c$, it is clear that if $\{a_p\}$ is a real sequence such that $|a_p - 1| \leqq |X_p - 1|$, $p = 1, 2, 3, \cdots$, then $A\{a_p\} \in S_c$. But A is not c.p.

THEOREM 3.4. *Suppose* X *is a real sequence which converges to* 1. *If* A *is c.p.o. the set of all real sequences which converge to* 1 *slower than* X, *then* A *is c.p.*

PROOF. Let $Y_p = |X_p - 1|^{1/2} + 1$ if $X_p \neq 1$, $Y_p = 1 + 1/p$ if $X_p = 1$, $p = 1, 2, 3, \cdots$. Then $\{Y_p\}$ converges to 1 slower than X. Thus $A\{Y_p\} \in S_c$. If $\{a_p\}$ is a real sequence which converges to 1 such that for each positive integer p, $|Y_p - 1| \leqq |a_p - 1|$, then $\{a_p\}$ converges slower than X, and so $A\{a_p\} \in S_c$. Hence by Theorem 3.2, A is c.p.

THEOREM 3.5. *If A is c.p.o. the set of all real sequences $\{a_p\}$ such that $|a_i| \geqq |a_{i+1}|$, $i=1, 2, 3, \cdots$, then A is c.p.*

PROOF. Let $a_p=1$ if $p=1$, $a_p=0$ if $p>1$. Then $\{A_{n1}\}=A\{a_p\}\in S_c$. Let $b_p=1$ if $p=1$ or 2, $b_p=0$ if $p>2$. Then $A\{b_p\}\in S_c$ and so $\{A_{n2}\}\in S_c$ since $A\{b_p\}=\{A_{p1}+A_{p2}\}$. Hence by an obvious induction, (1) of (1.1) holds. Clearly (2) of (1.1) holds since $A\{1\}\in S_c$. Suppose (3) of (1.1) does not hold. Let $A=B+iC$, where B and C are real triangular matrices ($B_{pq}=C_{pq}=0$ if $q>p$). Then either B or C fails to satisfy a condition comparable to (3) of (1.1); let us say B. Clearly B satisfies conditions comparable to (1) and (2) of (1.1). Using only a trivial modification of a proof of Hardy [1, p. 46], a real sequence $\{y_p\}$ can be found such that $\{|y_p|\}\in S_{c0\downarrow}$ and $B\{y_p\}$ contains a subsequence divergent to $+\infty$. Therefore (3) of (1.1) holds for A, and the proof is complete.

The proof of Theorem 3.5 can be modified slightly to prove the following theorem.

THEOREM 3.6. *If A is convergence preserving over the set of all sequences $\{a_p\}$ such that $|a_i|>|a_{i+1}|$, $i=1, 2, 3, \cdots$, then A is c.p.*

COROLLARY 3.6a. *If A is c.p.o. the set of all sequences $\{a_p\}$ such that $a_i\neq a_{i+1}$, $i=1, 2, 3, \cdots$, then A is c.p.*

THEOREM 3.7. *If A is convergence preserving over the set of all sequences whose terms are elements of the Cantor set, then A is c.p.*

PROOF. We note that $A\{1\}\in S_c$ since 1 is an element of the Cantor set. Suppose $\{a_p\}\in S_{c0+}$. Let k be a positive number such that $ka_p<1$, $p=1, 2, 3, \cdots$, and let $b_p=ka_p$, $p=1, 2, 3, \cdots$. Suppose n is a positive integer. Let $b_n=.b_{n1}b_{n2}b_{n3}\cdots$ in ternary fractional form. For each positive integer p, let $c_{np}=0$ if $b_{np}=0$ or 2, $c_{np}=1$ if $b_{np}=1$. Let $d_{np}=b_{np}-c_{np}$, $p=1, 2, 3, \cdots$, and denote $.c_{n1}c_{n2}c_{n3}\cdots$ and $.d_{n1}d_{n2}d_{n3}\cdots$ in ternary fractional form by c_n and d_n, respectively. We note that d_n and $2c_n$ are both elements of the Cantor set, and $a_n=k^{-1}(c_n+d_n)$. Since $\{c_p\}$ and $\{d_p\}$ both converge, we see that $A\{a_p\}\in S_c$. Hence by Observation 2.5, A is c.p.

REFERENCES

1. G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949.
2. Konrad Knopp, *Infinite sequences and series*, Dover, New York, 1956.

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