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University of Texas and
    University of Kentucky

NOTE ON A NONLINEAR EIGENVALUE PROBLEM

MARVIN SHINBROT[1]

1. In the theory of hydrodynamic stability, eigenvalue problems of the form

$$(1.1) \qquad Lu + \frac{1}{\lambda} Mu = \lambda u$$

arise [1, p. 430]. Here, L and M denote ordinary differential operators, the order of L exceeds that of M, and the boundary conditions are such that L is self-adjoint. One of the questions of interest is whether there exist eigenvalues of this problem and, if so, whether the corresponding eigenfunctions are complete.

Replacing λ by $1/\lambda$, it is easy to see that if L^{-1} exists, (1.1) is equivalent to

$$(1.2) \qquad \lambda u = Au + \lambda^2 Bu,$$

where $A = L^{-1}$ and $B = -L^{-1}M$ are compact, and A is symmetric. In this note, we shall consider the question of the completeness of the eigenfunctions of the following generalization of (1.2):

$$(1.3) \qquad \lambda u = Au + \lambda^\alpha B_\lambda u,$$

where $\alpha > 1$, A is compact and symmetric, and B_λ, which, as the notation indicates, may depend on λ, is merely bounded. More precise

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hypotheses on A and B will be specified below.

2. We shall operate entirely in a Hilbert space H. The symbols μ_n and v_n will be used consistently to denote the eigenvalues and corresponding eigenfunctions of A, in the usual order and with the usual normalization.

The subspace of H spanned by all but the first $N-1$ eigenfunctions of A will be denoted by H_N: $H_N = \{v_N, v_{N+1}, \cdots\}$. If infinitely many eigenfunctions of (1.3) exist, the space spanned by all but the first $N-1$ of them will be denoted by K_N.

It will be assumed that A satisfies the hypothesis

($\mathfrak{A}$). *A is a compact, symmetric operator with simple eigenvalues μ_n. Its eigenfunctions v_n span H. Moreover, if*

$$(2.1) \qquad \delta_n = \min_{j \neq n} |\mu_n - \mu_j|,$$

then

$$(2.2) \qquad \sum \left| \frac{\mu_n^\alpha}{\delta_n} \right|^2 < \infty.$$

The assumptions on B_λ are set forth in

($\mathfrak{B}$). *B_λ is an operator defined for all values of the parameter λ in the disk*

$$(2.3) \qquad \lambda \leqq \frac{\alpha}{\alpha - 1} \|A\|.$$

B_λ is uniformly bounded (by $\|B\|$, say) in the disk (2.3) and satisfies a Lipschitz condition

$$(2.4) \qquad \|B_\lambda - B_\mu\| \leqq L |\lambda - \mu|$$

there.

We shall prove the following

THEOREM. *Let ($\mathfrak{A}$) and ($\mathfrak{B}$) be satisfied and let α exceed unity. Then, there exist infinitely many eigenvalues λ_n and eigenfunctions u_n of (1.3). The sequence $\{\lambda_n\}$ approaches zero. Moreover, there is an integer N such that $K_N \supset H_N$ and, if x is any element of H, it can be represented in the form*

$$(2.5) \qquad x = \sum_{1}^{N-1} \xi_n v_n + \sum_{N}^{\infty} \xi_n u_n.$$

Finally, if $\|A\|$ and the series (2.2) are small enough,[2] N may be

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[2] The assumption on $\|A\|$ cannot be eliminated by normalization since the required size of $\|A\|$ depends on $\|B\|$ in just such a way that the normalization constant cancels out.

taken as unity, so that the sequence $\{u_n\}$ is complete in H.

Before proving this theorem, let us note here once and for all that the letter c, with or without subscripts, will always be used to denote a constant independent of n. Generally, it will denote different constants in different formulas.

3. It is easy to prove the existence of a sequence of real eigenvalues approaching zero, even if (2.2) is violated, if B_λ is compact and symmetric and if

$$(3.1) \qquad \|B\| \cdot \|A\|^{\alpha-1} \leqq \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha} .$$

Define a mapping of the *interval* (2.3) into the real line as follows. Choose a value of λ in (2.3) and, for any fixed n, let λ' denote the nth eigenvalue of the operator $A_\lambda = A + \lambda^\alpha B_\lambda$. Note that λ' exists since λ is fixed, while A_λ is compact and symmetric and does not depend on λ'. Also,

$$|\lambda'| \leqq \|A_\lambda\| \leqq \|A\| + |\lambda|^\alpha\|B\| \leqq \frac{\alpha}{\alpha-1}\|A\|,$$

by (3.1) and the choice of λ. Thus, the mapping takes (2.3) into itself and is also continuous if B_λ is. For each n, then, there exists a fixed point—that is, a value of λ_n such that A_{λ_n} has λ_n as one of its eigenvalues. These λ_n are obviously eigenvalues of (1.3).

Moreover, $\lambda_n \to 0$, since for each positive value of λ, however small, all but a finite number of the eigenvalues λ' are absolutely less than λ. If n is large enough, then, the mapping just defined takes the interval $[-\lambda, \lambda]$ into itself. Since λ was arbitrary, this shows that $\lambda_n \to 0$.

Although the hypothesis (2.2) was not needed in this argument, the assumptions on B_λ had to be increased. Also, and more important, the method just described does not yield enough information for anything to be said about completeness of the eigenfunctions. To draw the conclusion (2.5), an existence proof that includes certain estimates of λ_n and u_n is required. We now proceed to give such a proof.

Let b_γ denote the ball of radius γ in H: $b_\gamma = \{u : \|u\| \leqq \gamma\}$. For some γ, which will be determined later, select an element $u \in b_\gamma$. Fix n, and consider the equation

$$(3.2) \qquad \lambda' = \mu_n + \lambda^\alpha(B_\lambda u, v_n),$$

where λ is any number in a disk

$$(3.3) \qquad\qquad |\lambda| \leqq \frac{\alpha}{\alpha - 1} |\mu_n|.$$

B_λ is defined, since $|\mu_n| \leqq \|A\|$ for all n. Also,

$$|\lambda'| \leqq |\mu_n| + \frac{\gamma \|B\| \alpha^\alpha}{(\alpha - 1)^\alpha} |\mu_n|^\alpha,$$

so that λ' satisfies (3.3) if n is large enough.

Since the mapping (3.2) is also continuous, we can define λ_n as the largest fixed point that satisfies (3.3). For each $u \in b_\gamma$, we obtain in this way a sequence $\{\lambda_n\}$ satisfying

$$(3.4) \qquad\qquad \lambda_n = \mu_n + \lambda_n^\alpha (B_n u, v_n)$$

(where B_n has been written for B_{λ_n}) and

$$(3.5) \qquad\qquad |\lambda_n| \leqq c |\mu_n| \qquad\qquad (c > 1).$$

Now, define a mapping of b_γ into H by the equation

$$(3.6) \qquad u' = v_n + \sum_{j \neq n} \frac{\lambda_n^\alpha}{\mu_n - \mu_j} (B_n u - (B_n u, v_n) u, v_j) v_j.$$

At this point, we determine γ so that b_γ is mapped into itself. We have

$$\|u'\|^2 = 1 + \sum_{j \neq n} \left| \frac{\lambda_n^\alpha}{\mu_n - \mu_j} (B_n u - (B_n u, v_n) u, v_j) \right|^2$$

$$\leqq 1 + c \left| \frac{\mu_n^\alpha}{\delta_n} \right|^2 \sum_{j \neq n} |(B_n u - (B_n u, v_n) u, v_j)|^2,$$

by (2.1) and (3.5). Consequently,

$$\|u'\|^2 \leqq 1 + c \left| \frac{\mu_n^\alpha}{\delta_n} \right|^2 \cdot \|B_n u - (B_n u, v_n) u\|^2 \leqq 1 + c \left| \frac{\mu_n^\alpha}{\delta_n} \right|^2 \gamma^2 (1 + \gamma)^2,$$

where c is not only independent of n but also of γ. In view of the convergence hypothesis (2.2), we see that for any $\gamma > 1$ we can make $\|u'\| \leqq \gamma$ by choosing n large enough.

It will now be shown that (3.6) is a contraction, again if n is large enough. Let u_1 and u_2 be any two elements of b_γ, and, suppressing the index n for now, let λ_1 and λ_2 be the corresponding solutions of

(3.4). Both λ_1 and λ_2 satisfy (3.5), of course. If u_1' and u_2' denote the images of u_1 and u_2 under (3.6), we have, just as before,

$$\|u_2' - u_1'\| \leqq \frac{1}{\delta_n} \|\lambda_2^\alpha B_{\lambda_2} u_2 - \lambda_1^\alpha B_{\lambda_1} u_1\|$$

$$+ \frac{1}{\delta_n} \|\lambda_2^\alpha (B_{\lambda_2} u_2, v_n) u_2 - \lambda_1^\alpha (B_{\lambda_1} u_1, v_n) u_1\|$$

$$(3.7) \qquad \leqq \frac{1}{\delta_n} \|\lambda_2^\alpha B_{\lambda_2} u_2 - \lambda_1^\alpha B_{\lambda_1} u_1\| + \frac{|\lambda_2|^\alpha}{\delta_n} |(B_{\lambda_2} u_2, v_n)| \cdot \|u_2 - u_1\|$$

$$+ \frac{1}{\delta_n} \|u_1\| \cdot |\lambda_2^\alpha (B_{\lambda_2} u_2, v_n) - \lambda_1^\alpha (B_{\lambda_1} u_1, v_n)|$$

$$\leqq \frac{c_1}{\delta_n} \|\lambda_2^\alpha B_{\lambda_2} u_2 - \lambda_1^\alpha B_{\lambda_1} u_1\| + c_2 \frac{|\mu_n^\alpha|}{\delta_n} \|u_2 - u_1\|,$$

by (3.5).

According to (3.4),

$$(3.8) \qquad |\lambda_2 - \lambda_1| \leqq \|\lambda_2^\alpha B_{\lambda_2} u_2 - \lambda_1^\alpha B_{\lambda_1} u_1\|.$$

But

$$\|\lambda_2^\alpha B_{\lambda_2} u_2 - \lambda_1^\alpha B_{\lambda_1} u_1\|$$

$$\leqq |\lambda_2^\alpha - \lambda_1^\alpha| \cdot \|B_{\lambda_2} u_2\| + |\lambda_1^\alpha| \cdot \|(B_{\lambda_2} - B_{\lambda_1}) u_2\| + |\lambda_1^\alpha| \cdot \|B_{\lambda_1} (u_2 - u_1)\|$$

$$\leqq c_1 |\mu_n|^{\alpha-1} \cdot |\lambda_2 - \lambda_1| + c_2 |\mu_n|^\alpha \cdot |\lambda_2 - \lambda_1| + c_3 |\mu_n|^\alpha \cdot \|u_2 - u_1\|,$$

by (2.4) and (3.5). Therefore,

$$(3.9) \|\lambda_2^\alpha B_{\lambda_2} u_2 - \lambda_1^\alpha B_{\lambda_1} u_1\| \leqq c_1 |\mu_n|^{\alpha-1} \cdot |\lambda_2 - \lambda_1| + c_2 |\mu_n|^\alpha \cdot \|u_2 - u_1\|.$$

If n is large enough, (3.8) and (3.9) together give

$$\|\lambda_2^\alpha B_{\lambda_2} u_2 - \lambda_1^\alpha B_{\lambda_1} u_1\| \leqq c |\mu_n^\alpha| \cdot \|u_2 - u_1\|.$$

Then, by (3.7),

$$\|u_2' - u_1'\| \leqq c \left|\frac{\mu_n^\alpha}{\delta_n}\right| \cdot \|u_2 - u_1\|,$$

proving that (3.6) is a contraction if n is large enough.

With this fact established, we can go on to prove the first part of the theorem. For each n, (3.6) has (exactly) one fixed point, which we shall denote by u_n. Clearly,

$$(3.10) \qquad\qquad (u_n, v_n) = 1,$$

so that $(B_n u_n - (B_n u_n, v_n) u_n, v_n) = 0$, and

$$(\mu_n - A) u_n = \lambda_n^{\alpha} \sum_j (B_n u_n - (B_n u_n, v_n) u_n, v_j) v_j$$

$$= \lambda_n^{\alpha} B_n u_n - \lambda_n^{\alpha} (B_n u_n, v_n) u_n = \lambda_n^{\alpha} B_n u_n - (\lambda_n - \mu_n) u_n,$$

by (3.4). Therefore, u_n is an eigenfunction of (1.3), again if n is large enough. The corresponding eigenvalues approach zero by (3.5).

To prove the rest of the theorem, consider

$$\|u_n - v_n\|^2 = \sum_{j \neq n} \left| \frac{\lambda_n^{\alpha}}{\mu_n - \mu_j} (B_n u_n - (B_n u_n, v_n) u_n, v_j) \right|^2$$

$$\leq c \left| \frac{\mu_n^{\alpha}}{\delta_n} \right|^2 \cdot \|B_n u_n - (B_n u_n, v_n) u_n\|^2 \leq c \left| \frac{\mu_n^{\alpha}}{\delta_n} \right|^2,$$

since $u_n \in b_\gamma$. Thus, by (2.2),

$$(3.11) \qquad\qquad \sum_N^{\infty} \|u_n - v_n\|^2 < 1,$$

if N is large enough.

All the assumptions we have made on the size of n (or N) have amounted to assumptions that $|\mu_n|$, or $|\mu_n^{\alpha}/\delta_n|$, or $\sum_n^{\infty} |\mu_N^{\alpha}/\delta_n|^2$ are small enough. Since $|\mu_1|$, the largest eigenvalue of A, is equal to $\|A\|$, we can say that if both $\|A\|$ and $\sum_1^{\infty} |\mu_n^{\alpha}/\delta_n|^2$ are small enough, (3.11) will hold even with $N = 1$. In that case, let $\{a_n\}$ be any finite sequence of complex numbers. Then,

$$\left\| \sum a_n (u_n - v_n) \right\|^2 \leq \left(\sum \|u_n - v_n\|^2 \right) \left(\sum |a_n|^2 \right)$$

$$\leq \theta \sum |a_n|^2,$$

where $\theta < 1$. Therefore, by a theorem of Paley and Wiener [2], the sequence $\{u_n\}$ is complete in H. This proves the last sentence of the theorem.

To prove (2.5), a little more is needed. Choose N so that (3.11) holds, and define P as the projection operator on H_N, so that

$$Px = \sum_N^{\infty} (x, v_j) v_j$$

for any x in H. If $n \geq N$,

$$Pu_n - v_n\| = \|P(u_n - v_n)\| \leqq \|u_n - v_n\|,$$

so that

$$\sum_N^\infty \|Pu_n - v_n\|^2 < 1.$$

Since $\{Pu_n\}_N^\infty \subset H_N$, we conclude that this sequence is complete in H_N, again by the theorem of Paley and Wiener.

Let $x \in H$. Then,

$$x = Px + (I - P)x$$

$$= \sum_N^\infty \xi_n Pu_n + (I - P)x,$$

for some sequence $\{\xi_n\}$ of complex numbers. However, the last equation can be written

$$x = \sum_N^\infty \xi_n u_n + (I - P)\left(x - \sum_N^\infty \xi_n u_n\right),$$

and (2.5) follows from this. This completes the proof of the theorem.

4. We shall now consider, very briefly, an application to the differential equation (1.1) in the important case when

$$(4.1) \qquad\qquad Lu = \frac{d}{dx}\left(p\,\frac{du}{dx}\right).$$

Recalling that the parameter λ of (1.2) is the reciprocal of the λ in (1.1), we shall call μ an "eigenvalue" of L if $Lu = (1/\mu)u$. It is known [3, pp. 270 ff.] that as $n \to \infty$, the "eigenvalues" of an operator (4.1) are ultimately simple and are asymptotic to a multiple of $1/n^2$. Thus, $\mu_n \sim c/n^2$, so that $\delta_n \sim c/n^3$. Therefore,

$$\sum \left|\frac{\mu_n^2}{\delta_n}\right|^2 \leqq c \sum \frac{1}{n^2} < \infty,$$

and the theorem of §2 can be applied.

References

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University of Chicago