This completes the proof of the theorem.

Note by the editor: The referee points out that the result of Fuchs to which the author refers appears in a slightly more general form in E. Hille's book, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R. I., 1948, p. 487.

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NOTE ON AN EMBEDDING THEOREM OF ADYAN

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1. **Introduction.** Adyan [1] studied the problem of embedding a finitely presented semigroup in a group. He obtained a sufficient condition for the embeddability but his paper does not discuss the question of necessity. In this note we give a simple counter-example that shows the condition is not necessary.

2. **Adyan's embedding theorem.** Let S be a finitely presented semigroup, in particular the semigroup with generators $a_1, a_2, \cdots, a_n$ and defining relations

$$A_k = B_k; \qquad k = 1, 2, \cdots, m \tag{1}$$

where A_k, B_k are products of positive powers of the generators and no A_k, B_k is empty.

With each defining relation $A_k = B_k$ associate an unordered pair (a_{i_k}, a_{j_k}) where A_k has a_{i_k} as its first generator and B_k has a_{j_k} or vice versa. This unordered pair is called the left pair of the defining relation $A_k = B_k$. The right pair is defined similarly, using the last generator from A_k and from B_k.

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[1] This note was prepared while the author held a studentship of the National Research Council of Canada.

By the left graph of the system (1) we mean the graph determined by the set of all generators that occur in (1) and the set of all left pairs of these relations. We say that the system (1) has no left cycles if the corresponding left graph contains no cycles. The right graph and right cycles can be defined similarly. If the system (1) has no left or right cycles it is said to contain no cycles.

Adyan's embedding theorem is as follows:

THEOREM (ADYAN). *If the system of defining relations* (1) *of a semigroup S has no cycles, then S is isomorphic to a subsemigroup of the group G with defining relations* (1).

3. **A counter-example.** That the absence of cycles is not a necessary condition for embeddability is shown by the following example.

Let S be the semigroup with generators a_1, a_2, a_3, x_1, x_2 and defining relations

$$\begin{aligned} a_1x_1 &= a_2x_2 \\ a_2x_1 &= a_3x_2 \\ a_3x_1 &= a_1x_2. \end{aligned} \tag{2}$$

The system (2) clearly has a left cycle. We shall show that it is embeddable. In order to do this we make use of some results of Mal'cev [**2**; **3**].

An equation $A = B$ involving only non-negative powers of the generators of S is called a group consequence of S if it holds in the group whose defining relations are (2). Consider a set of group consequences of S. We shall denote such a set by $C(S)$ for convenience, although this is not unique for S. Let S' be the semigroup with the equations of $C(S)$ together with (2) as defining relations. If every group consequence of S' is also true in S' we say that $C(S)$ together with (2) is a complete set of group consequences of S.

An application of one of Mal'cev's lemmas [**3**, Lemma 2] shows that the set of all group consequences of S of the forms $a_ix_j = a_kx_m$, $a_i = a_j$, $x_i = x_j$ is a complete set of group consequences. We shall show that the only equations of this form that are group consequences are the equations (2) and identities.

We shall first show that the only equations $a_i = a_j$ or $x_i = x_j$ that are group consequences are identities. To do this we need only exhibit a group for which equations (2) hold and for which the a_i's are distinct and $x_1 \neq x_2$.

Consider $Z/(7)$, the multiplicative group of nonzero residue classes of integers modulo 7. If we take $a_1 \equiv 1$, $a_2 \equiv 4$, $a_3 \equiv 2$, $x_1 \equiv 1$, $x_2 \equiv 2$ we can verify that equations (2) hold.

Now consider the nonidentical equations of the form $a_i x_j = a_k x_m$ other than the equations (2). Of these an equation of the form $a_i x_j = a_i x_k$ cannot be a group consequence, for then $x_j = x_k$, contrary to what we have just proved. Similarly $a_i x_j = a_k x_j$ cannot be a group consequence. The only equations that remain to be considered are $a_1 x_1 = a_3 x_2$, $a_1 x_2 = a_2 x_1$, and $a_2 x_2 = a_3 x_1$. If $a_1 x_1 = a_3 x_2$ is a group consequence then, since $a_1 x_1 = a_2 x_2$ we have $a_2 x_2 = a_3 x_2$ in the group, which contradicts $a_2 \neq a_3$. Thus $a_1 x_1 = a_3 x_2$ is not a group consequence. Similarly it can be shown that $a_1 x_2 = a_2 x_1$ and $a_2 x_2 = a_3 x_1$ are not group consequences.

This completes the proof that (2) is a complete set of group consequences of S and hence that all the group consequences of S already hold in S. The embedding theorem of [2] as restated in [3] now shows that S can be embedded in a group.

We now have a semigroup S with a left cycle which is embeddable in a group. Thus the absence of cycles cannot be a necessary condition for embeddability.

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