potents in each semigroup was a finite discrete set. It might be of interest to know if there exists a semigroup $S=ESE$ which is compact connected, has a zero, is not acyclic and such that the set of idempotents is connected.

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# CONFORMAL VECTOR FIELDS IN COMPACT RIEMANNIAN MANIFOLDS

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1. **Introduction.** Let $V^n$ be a compact Riemannian manifold of dimension $n$ and of class $C^3$. Let $g_{ij}(x)$ of class $C^2$ be the coefficients of the fundamental metric which is assumed to be positive definite. Let $\Gamma^h_{ij}$ be the Christoffell symbol, $R_{ijhk}$ the curvature tensor and $R_{ij}$ the Ricci tensor.

Let $\phi$ be an arbitrary scalar invariant, $\xi^i$ an arbitrary vector field and $\xi_{i_1i_2\cdots i_p}$ an arbitrary anti-symmetric tensor field of order $p$, all of class $C^2$ in $V^n$. We shall make use of the following results obtained by S. Bochner and K. Yano [1, pp. 31, 51, 69]:

(1.1) $(\Delta\phi \geqq 0 \text{ everywhere in } V^n) \Rightarrow (\phi = \text{constant everywhere in } V^n).$

$$\int_{V^n} \xi^i{}_{,i}\, dv = 0. \tag{1.2}$$

$$\int_{V^n} (R_{ij}\xi^i\xi^j + \xi^i{}_{,j}\,\xi^j{}_{,i} - \xi^i{}_{,i}\,\xi^j{}_{,j})dv = 0. \tag{1.3}$$

$$\int_{V^n} (F\{\xi_{i_1i_2\cdots i_p}\} + \xi^{ii_2\cdots i_p,j}\,\xi_{ji_2\cdots i_p,i} - \xi^{ii_2\cdots i_p}{}_{,i}\,\xi^j{}_{i_2\cdots i_p,j})dv = 0 \tag{1.4}$$

where

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$$F\{\xi_{i_1 i_2 \cdots i_p}\} = R_{ij}\,\xi^{i i_2 \cdots i_p}\,\xi^{j}{}_{i_2 \cdots i_p} + \frac{p-1}{2} R_{ijkl}\,\xi^{i j i_3 \cdots i_p}\,\xi^{kl}{}_{i_3 \cdots i_p}$$

and $V^n$ in equations (1.2) to (1.4) is assumed to be orientable.

If a vector field $\xi^i$ defines a one-parameter continuous group of infinitesimal conformal transformations in $V^n$, the intrinsic derivative of $\xi_i dx^i/ds$ along any geodesic $x^i(s)$ depends only on the point and not on the direction of the geodesic passing through the point; that is

$$\xi_{i,j}\frac{dx^i}{ds}\frac{dx^j}{ds} = \frac{1}{n}\,\xi^i{}_{,i}. \tag{1.5}$$

Let $\xi^i$ be an arbitrary vector field which does not necessarily define a one-parameter continuous group of infinitesimal conformal transformations. Then, instead of (1.5), we have in general

$$\xi_{i,j}\frac{dx^i}{ds}\frac{dx^j}{ds} = f\left(x^i, \xi^i, \frac{dx^i}{ds}\right).$$

In the following, such $\xi^i$ will be associated with an arbitrary but fixed scalar field $\phi$ in a certain way to give a generalization of some well-known vector fields.

2. **Definition.** Let $\lambda^i$ be an arbitrary unit vector field. Consider $M$ defined by

$$M = \xi_{i,j}\lambda^i\lambda^j - \phi. \tag{2.1}$$

When $\xi^i$ defines a one-parameter continuous group of infinitesimal conformal transformations and $\phi = \xi^i{}_{,i}/n$, we have, by (1.5), $M = 0$. Hence it seems appropriate to call $M$ the *$\phi$-conformal measure* of $\xi^i$ with respect to $\lambda^i$.

Let $\lambda_\alpha|^i$, $\alpha = 1, 2, \cdots, n$, be $n$ mutually orthogonal unit vector fields in $V^n$ and $M_\alpha$ the $\phi$-conformal measures of $\xi^i$ with respect to them, that is

$$M_\alpha = \xi_{i,j}\,\lambda_\alpha|^i\,\lambda_\alpha|^j - \phi.$$

The mean of $M_\alpha$ is equal to

$$\frac{1}{n}\sum_{\alpha=1}^{n} M_\alpha = \frac{1}{n} g^{ij}\,\xi_{i,j} - \phi \tag{2.2}$$

which is evidently independent of the choice of the orthogonal ennuple $\lambda_\alpha|^i$. A vector field $\xi^i$ for which the mean (2.2) vanishes is called a *$\phi$-conformal vector field* in $V^n$ or simply a *conformal vector field* in $V^n$. To each $\phi$ there corresponds one conformal vector field. The definition leads immediately to the following necessary and sufficient condition

$$\xi^i{}_{,i} = n\phi. \tag{2.3}$$

Since a harmonic vector field and a Killing vector field satisfy (2.3) when $\phi = 0$, a conformal vector field may be considered as a generalization of them in this sense.

We intend to investigate properties of conformal vector fields in a compact Riemannian manifold $V^n$, particularly the global nonexistence of these fields in $V^n$. A generalization of our concept to tensor fields is given at the end of the paper.

3. **Properties.** By definition of the Laplacean (denoted by $\Delta$) of a scalar field, we obtain for a conformal vector field $\xi^i$

$$\Delta\xi^i{}_{,i} = g^{jk}\,\xi^i{}_{,ijk} = n\Delta\phi. \tag{3.1}$$

The tendency of a vector $v^i$ in a unit direction $a^i$ is defined as the projection of the vector $a^k v^i{}_{,k}$ in the direction of $a^i$. It is well known that the divergence of a vector in $V^n$ is the sum of tendencies of the vector for $n$ mutually orthogonal directions in $V^n$ and that of a unit vector in $V^2$ is numerically equal to the geodesic curvature of a curve normal to the vector [**2**, p. 422; **3**, p. 201]. Hence from (1.1) and (1.2) we have the following two theorems.

THEOREM 3.1. *If $\phi$ satisfies $\Delta\phi \geqq 0$ everywhere in a compact Riemannian manifold $V^n$, then the sum of tendencies of the $\phi$-conformal vector field for $n$ mutually orthogonal directions is constant throughout the manifold. If $\phi$ satisfies $\Delta\phi \geqq 0$ everywhere in a compact Riemannian manifold $V^2$, then the geodesic curvature of the orthogonal trajectories of the curves of the $\phi$-conformal unit vector field is constant throughout the manifold.*

THEOREM 3.2. *In a compact orientable Riemannian manifold $V^n$, there exists no $\phi$-conformal vector field with $\phi > 0$ or $\phi < 0$ everywhere in the manifold and therefore the divergence of a $\phi$-conformal vector field is a constant everywhere in $V^n$ if and only if the constant is zero. In a compact orientable Riemannian manifold $V^2$, the orthogonal trajectories of the curves of a $\phi$-conformal unit vector field form a family of geodesics in the manifold if and only if the divergence of the vector field is constant throughout.*

By (2.3) we may write (1.3) as

$$\int_{V^n} (\xi^i{}_{,j}\, \xi^j{}_{,i} + R_{ij}\, \xi^i\, \xi^j) dv = \int_{V^n} n^2 \phi^2 dv$$

and

$$\int_{V^n} (\xi^i{}_{,j}\, \xi^j{}_{,i} - n^2\phi^2) dv = - \int_{V^n} R_{ij}\, \xi^i\, \xi^j\, dv$$

which lead to the following two theorems:

THEOREM 3.3. *In a compact orientable Riemannian manifold $V^n$ there exists no $\phi$-conformal vector field $\xi^i$ which satisfies*

$$\int_{V^n} (R_{ij}\, \xi^i\, \xi^j + \xi^i{}_{,j}\, \xi^j{}_{,i}) dv \leqq 0$$

*unless we have $\phi = 0$ and then automatically the equality sign holds.*

THEOREM 3.4. *In a compact orientable Riemannian manifold with negative (positive) definite Ricci curvature throughout, there exists no $\phi$-conformal vector field other than zero vector field which satisfies*

$$\int_{V^n} (\xi^i{}_{,j}\, \xi^j{}_{,i} - n^2\phi^2) dv \leqq 0 \quad (\geqq 0).$$

The above theorem includes as special cases some results about global nonexistence of harmonic vector field and Killing vector field obtained by S. Bochner [1, pp. 37, 39].

4. **Generalization.** An anti-symmetric tensor field $\xi_{i_1 i_2 \cdots i_p}$ of order $p$ is a conformal Killing tensor field if and only if

$$(4.1) \qquad \xi_{i i_2 \cdots i_p, j} \frac{dx^i}{ds} \frac{dx^j}{ds} = \phi_{i_2 \cdots i_p}$$

where

$$\phi_{i_2 \cdots i_p} = \frac{1}{n} g^{ij} \xi_{i i_2 \cdots i_p, j}$$

is an anti-symmetric tensor of order $p-1$ [1, p. 73].

Denote by $\lambda_\alpha|^i$ $n$ mutually orthogonal unit vector fields, by $\phi_{i_2 i_3 \cdots i_p}$ an arbitrary but fixed anti-symmetric tensor field of order $p-1$, and by $M_\alpha|_{i_2 \cdots i_p}$ the following tensor fields

$$(4.2) \qquad M_\alpha|_{i_2 \cdots i_p} = \xi_{i i_2 \cdots i_p, j}\, \lambda_\alpha|^i\, \lambda_\alpha|^j - \phi_{i_2 \cdots i_p}.$$

The mean of (4.2) is equal to

$$\frac{1}{n}\sum_{\alpha=1}^{n} M_\alpha|_{i_2\cdots i_p} = \frac{1}{n}\xi_{ii_2\cdots i_p,j}\,g^{ij} - \phi_{i_2\cdots i_p} \tag{4.3}$$

which is independent of the choice of the orthogonal ennuple $\lambda_\alpha|^i$. An anti-symmetric tensor field $\xi_{i_1\cdots i_p}$ for which (4.3) vanishes is called a *$\phi$-conformal tensor field* in $V^n$. Thus $\xi_{i_1\cdots i_p}$ is a $\phi$-conformal tensor field in $V^n$ if and only if

$$g^{ij}\,\xi_{ii_2\cdots i_p,j} = n\phi_{i_2\cdots i_p}. \tag{4.4}$$

Obviously, a $\phi$-conformal tensor field includes a conformal Killing tensor field as a special case.

Substituting (4.4) into (1.4) gives

THEOREM 4.1. *In a compact orientable Riemannian manifold there exists no $\phi$-conformal tensor field of order $p$ which satisfies*

$$F\{\xi_{i_1\cdots i_p}\} + \xi^{ii_2\cdots i_p,j}\,\xi_{ji_2\cdots i_p,i} \leqq 0$$

*unless we have* $\phi_{i_2\cdots i_p} = 0$ *and then automatically the equality sign holds.*

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