

INVERTIBLE ISOTOPIES

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Introduction. The origin of this paper was the remark that “any isotopy of n -dimensional euclidean space obviously possesses a unique extension to an isotopy of the n -sphere.” This is true, but not obvious. J. W. Milnor showed me a proof using invariance of domain, and later Herman Gluck, using the results of R. Arens [1], demonstrated that an isotopy of any locally compact, locally connected, Hausdorff space X can be extended to an isotopy of the 1-point compactification $X \cup \{\infty\}$. In Theorem (2.1) we show that local connectedness is not necessary. Local compactness, however, is needed (see Example (1.1) and the first paragraph of §2).

1. Example and theorem. An *isotopy* of a topological space X is a collection $\{h_t\}$, $0 \leq t \leq 1$, of autohomeomorphisms of X such that the mapping $h: X \times [0, 1] \rightarrow X$ defined by $h(x, t) = h_t(x)$ is continuous. Associated with any isotopy $\{h_t\}$ is also the mapping $H: X \times [0, 1] \rightarrow X \times [0, 1]$ defined by $H(x, t) = (h_t(x), t)$. This mapping is one-to-

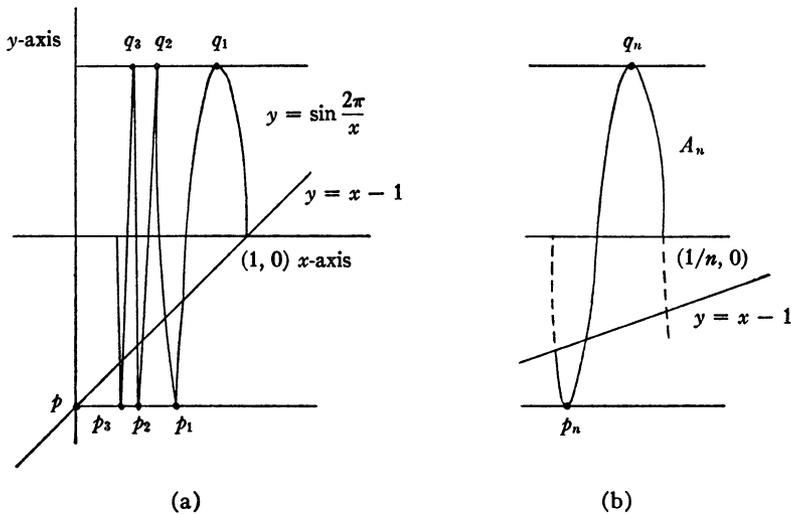


FIGURE 1

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one, onto, and continuous. We shall say that an isotopy $\{h_t\}$, $0 \leq t \leq 1$, is *invertible* if the collection $\{h_t^{-1}\}$, $0 \leq t \leq 1$, of inverse homeomorphisms is an isotopy. Obviously an isotopy $\{h_t\}$ is invertible if and only if the associated mapping H is a homeomorphism.

(1.1) EXAMPLE. Not every isotopy is invertible. Let X be the subspace of the xy -plane consisting of the point $p = (0, -1)$ plus all points (x, y) satisfying $y = \sin(2\pi/x)$, $0 < x \leq 1$. The space X is Hausdorff, but not locally compact. We define two sequences $\{p_n\}$ and $\{q_n\}$ of points in X as shown in Figure 1(a). A collection $\{h_t\}$, $0 \leq t \leq 1$, of autohomeomorphisms of X is defined as follows:

$$h_0 = h_{1/(2n-1)} = \text{identity}, \quad n = 1, 2, \dots$$

For any t in the closed interval $[1/(2n-1), 1/(2n+1)]$, h_t is the identity except on the open arc A_n running from the point $(1/n, 0)$ along X toward the y -axis and ending at the second point of intersection with the line $y = x - 1$, see Figure 1(b). This arc is mapped by h_t semi-linearly on itself so that during the interval $[1/(2n-1), 1/(2n+1)]$ the point q_n is slid over to p_n and back again without moving any points outside A_n . The timing is chosen so that

$$h_{1/2n}(q_n) = p_n, \quad n = 1, 2, \dots$$

The motion is illustrated in Figure 2 where the graph of $h_t|_{\overline{A}_n}$ is drawn for five different values of t .

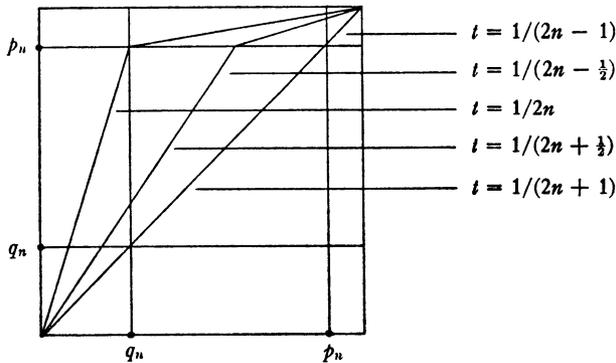


FIGURE 2

The mapping $h: X \times [0, 1] \rightarrow X$ defined by $h(x, t) = h_t(x)$ is obviously continuous except possibly at $(p, 0)$. However, every circular neighborhood of p of radius ≤ 1 is mapped into itself by every h_t . Hence h is continuous and $\{h_t\}$ is an isotopy of X . On the other hand,

the collection of inverse mappings h_t^{-1} is not an isotopy since $\lim_{n \rightarrow \infty} h_{1/2n}^{-1}(p_n) = \lim_{n \rightarrow \infty} q_n$ does not exist, but

$$h_0^{-1} \left(\lim_{n \rightarrow \infty} p_n \right) = h_0^{-1}(p) = p.$$

(1.2) THEOREM. *Every isotopy $\{h_t\}$ of a locally compact Hausdorff space X is invertible.*

PROOF. We must show that the mapping H defined by $H(x, t) = (h_t(x), t)$ is an autohomeomorphism of $X \times [0, 1]$. The only thing in doubt is the continuity of H^{-1} . We first prove the lemma:

For each $y \in X$, the mapping $t \rightarrow h_t^{-1}(y)$ is continuous. Choose $t_0 \in [0, 1]$, set $x = h_{t_0}^{-1}(y)$, and select an open neighborhood U of x such that the closure \bar{U} is compact. There then exists a compact neighborhood K of y such that $h_{t_0}^{-1}K \subset U$. We denote the complement of a set in X by means of a prime, and the interior by a small circle. Then,

$$h_{t_0}(U' \cap \bar{U}) \subset K', \quad h_{t_0}(x) \in K^\circ.$$

Since $U' \cap \bar{U}$ and x are compact, K' and K° are open, and H is continuous, there exists a positive number δ such that if $|t - t_0| < \delta$, then

$$(1) \quad h_t(U' \cap \bar{U}) \subset K', \quad h_t(x) \in K^\circ.$$

Assume $|t - t_0| < \delta$. Then $K \subset h_t(U \cup \bar{U}')$ and so

$$(2) \quad h_t^{-1}K \subset U \cup \bar{U}'.$$

Denote the closed interval between t_0 and t by I_t , and the connected set $h(x \times I_t)$ by C . Then $h_t(x) \in C$ and, by (1), $C \subset K^\circ$. Hence, by (2),

$$h_t^{-1}C \subset U \cup \bar{U}'.$$

The sets U and \bar{U}' are open and disjoint. Since $h_t^{-1}C$ is connected and $x \in U \cap h_t^{-1}C$, we conclude that

$$h_t^{-1}C \subset U.$$

But $y = h(x, t_0) \in C$. So we obtain

$$h_t^{-1}(y) \in U, \quad \text{if } |t - t_0| < \delta,$$

and the proof of the lemma is complete.

We now prove that H^{-1} is continuous. Consider $H(x_0, t_0) = (y_0, t_0)$, and let W be an open neighborhood of (x_0, t_0) in $X \times [0, 1]$. Choose U open in X so that \bar{U} is compact and

$$(x_0, t_0) \in U \times (t_0 - \epsilon, t_0 + \epsilon) \subset W.$$

Select in X a compact neighborhood K of y_0 such that $h_0^{-1}K \subset U$. Then, as before,

$$h_{t_0}(U' \cap \bar{U}) \subset K', \quad h_{t_0}(x_0) \in K^\circ.$$

This time we infer more. There exists a positive δ no bigger than ϵ , and an open subset V of U containing x_0 such that if $|t - t_0| < \delta$, then

$$h_t(U' \cap \bar{U}) \subset K', \quad h_t V \subset K^\circ.$$

As before, we have

$$h_t^{-1}K \subset U \cup \bar{U}', \quad \text{if } |t - t_0| < \delta.$$

The set $N = h_{t_0} V$ is an open neighborhood of y_0 , and is a subset of K° . We contend that

$$(3) \quad h_t^{-1}N \subset U, \quad \text{if } |t - t_0| < \delta.$$

Assume $|t - t_0| < \delta$, and consider an arbitrary element $h_t^{-1}(y)$, $y \in N$. Denote the closed interval between t_0 and t by I_t . By our lemma the set $C = \{h_s^{-1}(y) | s \in I_t\}$ is connected. Furthermore for an arbitrary $h_s^{-1}(y) \in C$, we have

$$h_s^{-1}(y) \in h_s^{-1}K^\circ \subset U \cup \bar{U}'.$$

So $C \subset U \cup \bar{U}'$. However, $h_{t_0}^{-1}(y)$ belongs to C and also to $V \subset U$. Since C is connected and U and \bar{U}' are disjoint open sets, we conclude that $C \subset U$. Hence $h_t^{-1}(y) \in U$, and (3) is established. But (3) implies

$$H^{-1}(N \times (t_0 - \delta, t_0 + \delta)) \subset U \times (t_0 - \epsilon, t_0 + \epsilon) \subset W,$$

so the proof of (1.2) is complete.

2. The 1-point compactification. It is easy to check that every autohomeomorphism $f: X \rightarrow X$ has a unique extension to an autohomeomorphism \bar{f} of the 1-point compactification $X \cup \{\infty\}$ given by $\bar{f}|X = f$ and $\bar{f}(\infty) = \infty$. Can a given isotopy $\{h_t\}$ of X be extended to an isotopy $\{\bar{h}_t\}$ of $X \cup \{\infty\}$? The answer in general is no. For in the example in (1.1), if $\bar{h}(x, t) = \bar{h}_t(x)$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \bar{h}(q_n, 1/2n) &= \lim_{n \rightarrow \infty} p_n = p, \\ \bar{h}\left(\lim_{n \rightarrow \infty} q_n, \lim_{n \rightarrow \infty} 1/2n\right) &= \bar{h}_0(\infty) = \infty. \end{aligned}$$

However, if X is locally compact and Hausdorff, the answer is yes.

(2.1) **THEOREM.** *Every isotopy $\{h_t\}$ of a locally compact Hausdorff space X has a unique extension to an isotopy of $\bar{X} = X \cup \{\infty\}$.*

PROOF. The question is whether or not the collection $\{\bar{h}_t\}$ of extended homeomorphisms is an isotopy. By Theorem (1.2), the isotopy $\{h_t\}$ is invertible; so the associated mapping H is a homeomorphism. Extend H to \bar{H} by setting

$$\bar{H}(\infty, t) = (\bar{h}_t(\infty), t) = (\infty, t).$$

To prove \bar{H} continuous, consider an open product neighborhood $U \times \Delta$ of (∞, t_0) . Then $U' = \bar{X} - U$ is a compact subset of X . Hence $K = H^{-1}(U' \times \bar{\Delta}) = \bar{H}^{-1}(U' \times \bar{\Delta})$ is compact and therefore closed in the Hausdorff space $\bar{X} \times [0, 1]$. The set $\bar{X} \times \Delta - K$ is open, contains (∞, t_0) , and

$$\bar{H}(\bar{X} \times \Delta - K) = \bar{X} \times \Delta - U' \times \bar{\Delta} = U \times \Delta.$$

So \bar{H} is continuous. If π_1 is the projection of $\bar{X} \times [0, 1]$ on \bar{X} , the composition $\bar{h} = \pi_1 \bar{H}$ is also continuous, and it follows that $\{\bar{h}_t\}$ is an isotopy.

3. The ideal compactification. Let X be connected, locally connected, locally compact, Hausdorff, and separable. The *ideal compactification* of Freudenthal [4; 5] is a connected, locally connected, compact, separable, Hausdorff space Y such that X is dense, open, and locally connected in Y (see [3, p. 245]) and the set $Y - X$ is 0-dimensional. Furthermore, these properties characterize the compactification (see [3, p. 249; 5, p. 276]).

(3.1) *Every autohomeomorphism f of X has a unique extension to an autohomeomorphism f of the ideal compactification Y .*

PROOF. Let $i: X \rightarrow Y$ be the inclusion mapping. In the terminology of Kelley [6] the compactification is the pair (i, Y) . Since the pair (if, Y) also satisfied the characteristic properties of the ideal compactification, it is an equivalent compactification. Hence there exists an autohomeomorphism \bar{f} of Y such that $\bar{f}i = if$. The extension \bar{f} is unique because Y is Hausdorff and X is dense in Y .

(3.2) THEOREM. *Every isotopy $\{h_t\}$ of X has a unique extension to an isotopy of the ideal compactification Y .*

PROOF. We must prove that the collection $\{\bar{h}_t\}$ of extended homeomorphisms is an isotopy of Y . By Theorem (1.2) the mapping H associated with $\{h_t\}$ is a homeomorphism, and we define the extension \bar{H} by

$$\bar{H}(x, t) = (\bar{h}_t(x), t).$$

As in the proof of (2.1), it suffices to prove \bar{H} continuous. Let

$p \in Y - X$ and $\bar{H}(p, t_0) = (q, t_0)$, and consider an arbitrary neighborhood W of (q, t_0) . Since $q \in Y - X$ which is 0-dimensional, there is an open neighborhood U of q in Y and an open interval Γ about t_0 such that $U \times \Gamma \subset W$ and $\dot{U} = \bar{U} - U \subset X$. The boundary \dot{U} is closed and therefore compact. Hence $K = H^{-1}(\dot{U} \times \bar{\Gamma})$ is compact and therefore closed in $Y \times [0, 1]$. Choose an open product neighborhood $V \times \Delta$ about (p, t_0) such that

$$V \times \Delta \subset Y \times [0, 1] - K,$$

Δ is an open subinterval of Γ ,

$$\bar{h}_{t_0} V \subset U,$$

$V \cap X$ is connected (X is locally connected in Y).

Since $V \cap X \times \Delta \subset X \times \Gamma - H^{-1}(\dot{U} \times \bar{\Gamma})$, we have

$$H(V \cap X \times \Delta) \subset X \times \Gamma - \dot{U} \times \bar{\Gamma} \subset U \cup \bar{U}' \times \Gamma.$$

The image $H(V \cap X \times \Delta)$ is connected, the sets U, \bar{U}' are open and disjoint, and h_{t_0} maps points of $V \cap X$ into U . We conclude that

$$H(V \cap X \times \Delta) \subset U \times \Gamma.$$

Consider finally any point $(r, t) \in (V \cap (Y - X)) \times \Delta$. Suppose $\bar{h}_t(r) \in \bar{U}'$ (it belongs either to U to \bar{U}'). Since \bar{h}_t is continuous and X is dense in Y , there then exists a point of $V \cap X$ which is also mapped by \bar{h}_t , and so by h_t , into \bar{U}' . This is impossible. We conclude that

$$\bar{H}(V \times \Delta) \subset U \times \Gamma \subset W,$$

and the proof is complete.

4. Conclusion. In this section X is always a locally compact Hausdorff space. The compact-open topology on the set $A(X)$ of all autohomeomorphisms of X defines a topological space which we denote by $A(X)_{\infty}$. If $\{h_t\}$, $0 \leq t \leq 1$, is an isotopy of X , then the mapping $h^*: [0, 1] \rightarrow A(X)_{\infty}$, defined by $h^*(t) = h_t$, is continuous. Conversely, any path in $A(X)_{\infty}$ defines an isotopy. (See R. H. Fox [2].) In [1] Arens considers what he calls the g -topology on $A(X)$, which is larger (more open sets) than the compact-open topology. For any two subsets $F, W \subset X$, let (F, W) denote the set of all $f \in A(X)$ such that $fF \subset W$. The g -topology is defined by taking for a sub-basis all sets (F, W) where F is closed, W is open, and either F or $X - W$ is compact. In the group $A(X)_{\infty}$ multiplication (composition) is continuous, but the operation $f \rightarrow f^{-1}$ generally is not. Arens proves that the g -topology is the smallest admissible topology making $A(X)$ a topological group. He also proves that if X is locally connected, then

$A(X)_{\infty} = A(X)_{\theta}$. It is easy to verify that an invertible isotopy is equivalent to a path $[0, 1] \rightarrow A(X)_{\theta}$. Combining the last two facts, we obtain Gluck's proof of Theorem (1.2) for a locally connected X . Arens also gives an example of a locally compact Hausdorff space for which the co and g topologies are *not* equal, or equivalently, the identity mapping $A(X)_{\infty} \rightarrow A(X)_{\theta}$ is not continuous. In view of Aren's example and theorem (1.2), we have the interesting result that for a locally compact Hausdorff space X and path $[0, 1] \xrightarrow{h^*} A(X)_{\infty}$, the composition $[0, 1] \xrightarrow{h^*} A(X)_{\infty} \xrightarrow{\text{id}} A(X)_{\theta}$ is necessarily continuous even though $A(X)_{\infty} \rightarrow A(X)_{\theta}$ is not. Actually, as Gluck observed, this result remains true if the path h^* is replaced by any continuous mapping $h^*: Y \rightarrow A(X)_{\infty}$ of an arbitrary locally connected topological space Y . The reason is that Theorem (1.2) remains true and the same proof works if in the definition of isotopy the interval $[0, 1]$ is replaced by Y . Thus the possible discontinuities of $\text{id}: A(X)_{\infty} \rightarrow A(X)_{\theta}$ cannot be detected by any continuous mapping of a locally connected space into $A(X)_{\infty}$.

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