SHORTER NOTES

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A NOTE ON FREE ALGEBRAS

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Let $\boldsymbol{K}$ be a class of abstract algebras of some fixed type. An algebra A is called a $\boldsymbol{K}$-free algebra if (1) $A\in\boldsymbol{K}$, and (2) A contains a subset X (called a system of free generators of A) such that X generates A, and every mapping of X into an algebra $B\in\boldsymbol{K}$ can be extended to a homomorphism of A into B. It is trivial to show that two $\boldsymbol{K}$-free algebras with the same cardinal number of free generators are isomorphic.

It has been known for some time that the free α-representable Boolean algebras (that is, free α-homomorphs of α-fields of sets) are α-fields (see [**2**, p. 107], and the references given there). Recently, Horn proved that the free α-representable lattices (free α-homomorphs of α-rings of sets) are α-rings of sets (see [**1**]). The purpose of this note is to point out that both of these results are included in a general theorem whose proof is elementary.

THEOREM. *Let* $\mathbf{L}$ *be a class of abstract algebras of some fixed type. Assume that every subalgebra of an algebra in* $\mathbf{L}$ *is in* $\mathbf{L}$, *and that every algebra isomorphic to an algebra of* $\mathbf{L}$ *is in* $\mathbf{L}$. *Let* $\boldsymbol{M}$ *be the class of all algebras which are homomorphic images of algebras in* $\mathbf{L}$. *Then an algebra* A *is* $\boldsymbol{M}$*-free if and only if it is* $\mathbf{L}$*-free.*

PROOF. Suppose that A is $\boldsymbol{M}$-free. Let X be a system of free generators of A. Since every algebra of $\boldsymbol{M}$ is a homomorph of an algebra in $\boldsymbol{L}$, there is an algebra $B\in\boldsymbol{L}$ and a homomorphism h of B onto A. By the axiom of choice, there is a mapping g of X into B such that $h(g(x))=x$ for all $x\in X$. Because A is $\boldsymbol{M}$-free and $B\in\boldsymbol{L}\subseteq\boldsymbol{M}$, it is possible to extend g to a homomorphism of A onto a subalgebra of B. Since $\{z\in A\,|\,h(g(z))=z\}$ is a subalgebra of A containing X, and X generates A, it follows that g is one-to-one. Thus, A is isomorphic to a subalgebra of an algebra in $\boldsymbol{L}$. Therefore, $A\in\boldsymbol{L}$ and consequently A is $\boldsymbol{L}$-free. The converse is a known fact which is easily proved by observing that any mapping from a system X of free generators of an $\boldsymbol{L}$-free algebra A to an algebra $C\in\boldsymbol{M}$ can be lifted to an algebra

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$B \in L$ which has C as a homomorphic image. Since A is L-free, the lifted mapping extends to a homomorphism of A into B which composes with the homomorphism of B onto C to give an extension of the original mapping.

References

1. A. Horn, *On α-homomorphic images of α-rings of sets*, Fund. Math. **51** (1962), 259–266.
2. R. Sikorski, *Boolean algebras*, Springer, Berlin, 1960.

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