THE EULER ϕ FUNCTION FOR GENERALIZED INTEGERS

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1. Introduction. In a previous paper [1] generalized integers were defined as follows. Suppose there is given a finite or infinite sequence $\{p\}$ of real numbers (generalized primes) such that $1<p_1<p_2< \cdots$. Form the set $\{l\}$ of all possible p-products, i.e., products $p_1^{v_1}p_2^{v_2} \cdots$ where $v_1, v_2, \cdots$, are integers $\geqq 0$ of which all but a finite number are 0. Call these numbers generalized integers and suppose that no two generalized integers are equal if their v's are different. Then arrange $\{l\}$ as an increasing sequence: $1=l_1<l_2< \cdots$.

DEFINITIONS. Let $[x]$ = the number of generalized integers $\leqq x$ where x is any real number. If k is a positive integer and l_n a generalized integer, denote by $\phi_k(l_n)$ the number of generalized integers l_m in the set $1, l_2, l_3, \cdots, l_n$, for which the greatest common divisor (l_m, l_n) is kth power free. Then $\phi_1(l_n)$ is the Euler ϕ function for generalized integers defined by

$\phi_1(l_n)$ = the number of generalized integers $\leqq l_n$ which are prime to l_n, $=\phi(l_n)$ (say).

Let $\mu(l_n)$ be the Möbius function for generalized integers defined by $\mu(l_n)=0$ if l_n has a square factor; $\mu(l_n)=(-1)^k$, where k denotes the number of prime divisors of l_n and l_n has no square factor; $\mu(1)=1$. Then

$$(1.1)\quad \phi_k(l_n^k) = \sum_{d\delta=l_n} \mu(d)[\delta^k] \qquad (d \text{ and } \delta \text{ are generalized integers}).$$

This is proved in [1] for $k=1$, and the proof for $k>1$ is similar.

The following assumption will be used throughout this paper:—

$$(1.2)\qquad [x] = x + R(x) \quad \text{where} \quad R(x) = O(x^\alpha) \quad \text{and} \quad 0 < \alpha < 1.$$

The case $\alpha=0$ will be considered separately at the end of this paper.

Eckford Cohen [2] has written on arithmetical functions of a greatest common divisor using the Euler ϕ function as a particular example. The aim of this paper is to examine some of his results in the light of generalized integers. It will be noted that only the method of construction of generalized integers is needed for finite counting processes but an assumption of some sort, in this paper (1.2), is needed to find orders of magnitude.

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2. The generalized Zeta function, supplementary estimates and various identities. Define

$$\zeta(s) = \sum_{n=1}^{\infty} l_n^{-s} \qquad (s > 1).$$

Then it is proved in [3], using an assumption equivalent to (1.2) that

$$\zeta(s) = \prod_{r=1}^{\infty} \frac{1}{1 - p_r^{-s}} \cdot$$

(Incidentally, assumption (1.2) ensures that the number of generalized primes is infinite.) Hence

$$\frac{1}{\zeta(s)} = \prod_{r=1}^{\infty} (1 - p_r^{-s}) = \sum_{n=1}^{\infty} \mu_n l_n^{-s}. \tag{2.1}$$

Abel's transformation, in the following form, will be used to give some necessary estimates. Suppose $\{\lambda_n\}$ and $\{a_n\}$ are given with $\lambda_1 \leqq \lambda_2 \leqq \cdots$; $\lambda_n \to \infty$. Let $A(x) = \sum_{\lambda_n \leq x} a_n$. Suppose $\psi(x)$ has a continuous derivative $\psi^1(x)$ for all x involved. Then

$$\sum_{\lambda_n \leq x} a_n \psi(\lambda_n) = A(x)\psi(x) - \int_{\lambda_1}^{x} A(x)\psi^1(x)dx.$$

Using (1.2) and this transformation, the following results may be obtained.

$$\sum_{l_n \leq x} \frac{1}{l_n^{\beta}} = \frac{x^{1-\beta}}{1-\beta} + \gamma_\beta + O(x^{\alpha-\beta}) \qquad \beta \neq 1; \beta \neq \alpha, \tag{2.21}$$

$$\sum_{l_n \leq x} \frac{1}{l_n} = \log x + \gamma_1 + O(x^{\alpha-1}), \tag{2.22}$$

$$\sum_{l_n > x} \frac{1}{l_n^{\beta}} = \zeta(\beta) - \sum_{l_n \leq x} \frac{1}{l_n^{\beta}} = O\left(\frac{1}{x^{\beta-1}}\right) \qquad \text{if } \beta > 1, \tag{2.23}$$

$$\sum_{l_n \leq x} \frac{\log l_n}{l_n^2} = \delta_2 - \frac{\log x}{x} - \frac{1}{x} + O(x^{\alpha-2}\log x), \tag{2.24}$$

$$\sum_{l_n \leq x} \frac{1}{l_n^{\alpha}} = \frac{x^{1-\alpha}}{(1-\alpha)} + O(\log x), \tag{2.25}$$

$$\sum_{l_n \leq x} \frac{\log l_n}{l_n^{1+\alpha}} = \delta_{1+\alpha} - \frac{\log x}{\alpha x^{\alpha}} - \frac{1}{x^{\alpha}} + O\left(\frac{\log x}{x}\right), \tag{2.26}$$

$$\sum_{l_n > x} \frac{\log l_n}{l_n^2} = O\left(\frac{\log x}{x}\right), \tag{2.27}$$

$$\sum_{l_n \leq x} \frac{R(l_n)}{l_n^\beta} = c_\beta + O(x^{\alpha+1-\beta}), \tag{2.28}$$

where γ_β, δ_β and c_β are constants.

Define

$$G(x) = \sum_{l_n \leq x} g(l_n), \qquad H(x) = \sum_{l_n \leq x} h(l_n), \qquad f(l_n) = \sum_{d\delta = l_n} g(d)h(\delta)$$

where $g(l_n)$ and $h(l_n)$ are arithmetical functions. Then

LEMMA 2.1.

$$\sum_{l_n \leq x} f(l_n) = \sum_{l_n \leq x} g(l_n) H\left(\frac{x}{l_n}\right) = \sum_{l_n \leq x} h(l_n) G\left(\frac{x}{l_n}\right). \tag{2.3}$$

LEMMA 2.2. *For all* x_1, x_2 *satisfying* $0 < x_1 \leqq x$, $0 < x_2 \leqq x$, $x_1 x_2 = x$,

$$\sum_{l_n \leq x} f(l_n) = \sum_{l_n \leq x_1} g(l_n) H\left(\frac{x}{l_n}\right) + \sum_{l_n \leq x_2} h(l_n) G\left(\frac{x}{l_n}\right) - G(x_1) H(x_2). \tag{2.4}$$

Lemmas 2.1 and 2.2 may be proved in exactly the same way as in [2].

LEMMA 2.3. *Define*

$$f_k(l_n) = \sum_{d\delta = l_n} g(d)[\delta^k]$$

where $g(l_n)$ *is a bounded arithmetical function and* k *a real number. Also define*

$$L(s, g) = \sum_{n=1}^{\infty} \frac{g(l_n)}{l_n^s} \qquad (s > 1),$$

and let $L^1(s, g)$ *denote the derivative of* $L(s, g)$. *Then if* $k > 0$,

$$\sum_{l_n \leq x} f_k(l_n) = \frac{L(k+1, g)}{k+1} \cdot x^{k+1} + O(x) + O(x^{\alpha+k}) + O(x^{1+\alpha k}), \tag{2.5}$$

$$\sum_{l_n \leq x} \frac{f_1(l_n)}{l_n^t} = \frac{L(2, g) x^{2-t}}{(2-t)} + O(x^{1+\alpha-t}) + O(1), \quad t \neq 2; t \neq 1 + \alpha, \tag{2.6}$$

$$\sum_{l_n \leq x} \frac{f_1(l_n)}{l_n^2} = L(2, g)(\log x + C_1) + L^1(2, g) + O(x^{\alpha-1}), \tag{2.7}$$

$$C_1 = \gamma_1 + c_2,$$

$$\sum_{l_n \le x} \frac{f_1(l_n)}{l_n^{1+\alpha}} = \frac{L(2, g)x^{1-\alpha}}{1-\alpha} + O(\log x). \tag{2.8}$$

PROOF. For all values of k and t,

$$\begin{aligned} \sum_{l_n \le x} \frac{f_k(l_n)}{l_n^t} &= \sum_{l_n \le x} \frac{1}{l_n^t} \sum_{d\delta = l_n} g(d)[\delta^k] = \sum_{d \le x} \frac{g(d)}{d^t} \sum_{\delta \le x/d} \frac{[\delta^k]}{\delta^t} \\ &= \sum_{d \le x} \frac{g(d)}{d^t} \sum_{\delta \le x/d} \left(\frac{1}{\delta^{t-k}} + \frac{R(\delta^k)}{\delta^t} \right) \qquad \text{from (1.2).} \end{aligned} \tag{2.9}$$

However,

$$\sum_{\delta \le x/d} \left(\frac{1}{\delta^{t-k}} + \frac{R(\delta^k)}{\delta^t} \right) = \begin{cases} \log \dfrac{x}{d} + \gamma_1 + c_2 + O\left(\dfrac{x}{d}\right)^{\alpha-1} & \text{for } k = 1, t = 2 \\ \zeta(t-1) + c_t + O\left(\dfrac{x}{d}\right)^{2-t} + O\left(\dfrac{x}{d}\right)^{1-t+\alpha} & \text{for } k = 1, t > 2 \end{cases} \tag{2.10}$$

from (2.22), (2.23) and (2.28). Also

$$\sum_{\delta \le x/d} \left(\frac{1}{\delta^{t-k}} + O\left(\frac{1}{\delta^{t-\alpha k}}\right)\right) = \begin{cases} \dfrac{x^{1+k}}{(1+k)} \cdot \dfrac{1}{d^{1+k}} + O\left(\dfrac{x}{d}\right)^{\alpha+k} + O\left(\dfrac{x}{d}\right)^{1+\alpha k} & \text{for } k > 0, t = 0 \\ \dfrac{x^{2-t}}{(2-t)} \cdot \dfrac{1}{d^{2-t}} + O(1) + O\left(\dfrac{x}{d}\right)^{1+\alpha-t} & \text{for } k = 1, t \ne 2, t \ne 1+\alpha \\ \dfrac{x^{1-\alpha}}{(1-\alpha)} \cdot \dfrac{1}{d^{1-\alpha}} + O\left(\log \dfrac{x}{d}\right) & \text{for } k = 1, t = 1+\alpha, \end{cases} \tag{2.11}$$

from (2.21), (2.25) and (2.22). Substituting from (2.11) in (2.9) gives

$$\begin{aligned} \sum_{l_n \le x} f_k(l_n) &= \sum_{d \le x} g(d) \left(\frac{x^{1+k}}{(1+k)} \cdot \frac{1}{d^{1+k}} + O\left(\frac{x}{d}\right)^{\alpha+k} + O\left(\frac{x}{d}\right)^{1+\alpha k} \right) \\ &= \frac{x^{1+k}}{(1+k)} L(1+k, g) + O(x) + O(x^{\alpha+k}) + O(x^{1+\alpha k}) \end{aligned}$$

from (2.23) and (2.21), (2.22) being used if $\alpha + k = 1$;

$$\sum_{l_n \leq x} \frac{f_1(l_n)}{l_n^t}$$

$$= \sum_{d \leq x} \frac{g(d)}{d^t}\left(\frac{x^{2-t}}{(2-t)} \cdot \frac{1}{d^{2-t}} + O(1) + O\left(\frac{x}{d}\right)^{1+\alpha-t}\right), \qquad t \neq 2; t \neq 1+\alpha$$

$$= \frac{x^{2-t}}{(2-t)} \sum_{d \leq x} \frac{g(d)}{d^2} + O\left(\sum_{d \leq x} \frac{g(d)}{d^t}\right) + O\left(x^{1+\alpha-t} \sum_{d \leq x} \frac{g(d)}{d^{1+\alpha}}\right)$$

$$= \frac{x^{2-t}}{(2-t)} L(2, g) + O(x^{1-t}) + O(1) + O(x^{1+\alpha-t})$$

from (2.23) and (2.21), (2.22) being used if $t=1$ since $\log x = O(x^\alpha)$;

$$\sum_{l_n \leq x} \frac{f_1(l_n)}{l_n^{1+\alpha}} = \sum_{d \leq x} \frac{g(d)}{d^{1+\alpha}}\left(\frac{x^{1-\alpha}}{(1-\alpha)} \frac{1}{d^{1-\alpha}} + O\left(\log \frac{x}{d}\right)\right)$$

$$= \frac{x^{1-\alpha}}{(1-\alpha)} L(2, g) + O(x^{-\alpha}) + O(\log x) + O(1)$$

from (2.23), (2.21) and (2.26). This proves (2.5), (2.6) and (2.8). Substituting from (2.10) in (2.9) gives

$$\sum_{l_n \leq x} \frac{f_1(l_n)}{l_n^2} = \sum_{d \leq x} \frac{g(d)}{d^2}\left(\log \frac{x}{d} + \gamma_1 + c_2 + O\left(\frac{x}{d}\right)^{\alpha-1}\right)$$

$$= (\log x + \gamma_1 + c_2) \sum_{d \leq x} \frac{g(d)}{d^2} - \sum_{d \leq x} \frac{g(d) \log d}{d^2}$$

$$+ O\left(x^{\alpha-1} \sum_{d \leq x} \frac{g(d)}{d^{1+\alpha}}\right)$$

$$= (\log x + \gamma_1 + c_2) L(2, g) + L^1(2, g) + O\left(\frac{\log x}{x}\right)$$

$$+ O\left(\frac{\log x}{x}\right) + O(x^{\alpha-1}),$$

from (2.23), (2.27) and (2.21). This proves (2.7) with $C_1 = \gamma_1 + c_2$ and so completes the proof of Lemma 2.3.

3. **The Euler ϕ function for generalized integers.** In Lemma 2.3 put $g(l_n) = \mu(l_n)$. Then $f_k(l_n) = \phi_k(l_n^k)$ and $L(s, \mu) = 1/\zeta(s)$, from (1.1) and (2.1).

We therefore obtain from Lemma 2.3

$$(3.1)\quad \sum_{l_n\leq x}\phi_k(l_n^k) = \frac{1}{(k+1)}\cdot\frac{x^{k+1}}{\zeta(k+1)} + O(x^{1+\alpha k}),\qquad k\geqq 1;$$

$$(3.2)\quad \sum_{l_n\leq x}\frac{\phi(l_n)}{l_n^t} = \frac{x^{2-t}}{(2-t)\zeta(2)} + O(x^{1+\alpha-t}) + O(1),\qquad \text{for } t\neq 2 \text{ and } t\neq 1+\alpha;$$

$$(3.3)\quad \sum_{l_n\leq x}\frac{\phi(l_n)}{l_n^2} = \frac{1}{\zeta(2)}\left(\log x - \frac{\zeta^1(2)}{\zeta(2)} + C_1\right) + O(x^{\alpha-1});$$

$$(3.4)\quad \sum_{l_n\leq x}\frac{\phi(l_n)}{l_n^{1+\alpha}} = \frac{x^{1-\alpha}}{(1-\alpha)\zeta(2)} + O(\log x).$$

Again, for $t>2$, we have from (2.9) and (2.10)

$$\begin{aligned}\sum_{l_n\leq x}\frac{\phi(l_n)}{l_n^t} &= \sum_{d\leq x}\frac{\mu(d)}{d^t}\sum_{\delta\leq x/d}\left(\frac{1}{\delta^{t-1}}+\frac{R(\delta)}{\delta^t}\right)\\ &= \sum_{d\leq x}\frac{\mu(d)}{d^t}\left(\zeta(t-1)+c_t+O\left(\frac{x}{d}\right)^{2-t}+O\left(\frac{x}{d}\right)^{1-t+\alpha}\right)\\ &= \frac{\zeta(t-1)+c_t}{\zeta(t)}+O\left(\frac{1}{x^{t-1}}\right)+O(x^{2-t})+O(x^{1-t+\alpha})\end{aligned}$$

from (2.23) and (2.21). This proves

$$(3.5)\qquad \sum_{l_n\leq x}\frac{\phi(l_n)}{l_n^t} = \left(\frac{\zeta(t-1)+c_t}{\zeta(t)}\right)+O\left(\frac{1}{x^{t-2}}\right);\qquad t>2.$$

LEMMA 3.1. *If $h(l_n)$ is an arbitrary arithmetical function, then*

$$(3.6)\qquad \sum_{l_p,l_q\leq x} h((l_p,l_q)) = 2\sum_{l_n\leq x} h(l_n)\Phi(x/l_n) - \sum_{l_n\leq x} h(l_n),$$

where (l_p, l_q) is the greatest common divisor of l_p and l_q and $\Phi(x) = \sum_{l_n\leq x}\phi(l_n)$.

PROOF. Let $Q(x)$ denote the number of ordered pairs of generalized integers $l_p, l_q\leqq x$ such that $(l_p, l_q)=1$. Then

$$\sum_{l_p,l_q\leq x} h((l_p,l_q)) = \sum_{l_n\leq x} Q(x/l_n)h(l_n).$$

But $Q(x)=2\Phi(x)-1$. Hence (3.6) follows immediately.

4. The average order of $f_k((l_p, l_q))$ where $f_k(l_n)=\sum_{d\delta=l_n} g(d)[\delta^k]$.

THEOREM 4.1. *Let $g(l_n)$ be bounded. Then in the case $k>1$,*

$$\sum_{l_p, l_q \leq x} f_k((l_p, l_q)) = \frac{L(k+1, g)}{(k+1)\zeta(k+1)} (2\zeta(k) + 2c_{k+1} - \zeta(k+1))x^{k+1} + O(x^2) + O(x^{\alpha+k});$$

and in the case $k=1$,

$$\sum_{l_q, l_q \leq x} f_1((l_p, l_q)) = \frac{x^2}{\zeta(2)} \left\{ L(2, g) \left(\log x + 2C_1 - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{\zeta^1(2)}{\zeta(2)} \right) + L^1(2, g) \right\} + O(x^{(3+\alpha)/2}).$$

PROOF. The proof of Theorem 4.1 parallels exactly the work of Eckford Cohen in [2] as follows: Let $F_k(l_n)$ denote the summatory function of $f_k(l_n)$. For all $k \geqq 1$, it follows from Lemma 3.1 that

$$\sum_{l_p, l_q \leq x} f_k((l_p, l_q)) = 2 \sum_{l_n \leqq x} f_k(l_n)\Phi(x/l_n) - F_k(x) = 2 \sum_1 - \sum_2. \tag{4.1}$$

Case 1. ($k > 1$).

$$\sum_1 = \sum_{l_n \leq x} f_k(l_n)\Phi(x/l_n) = \sum_{l_n \leq x} \phi(l_n)F_k(x/l_n) \qquad \text{from Lemma 2.1}$$

$$= \sum_{l_n \leq x} \phi(l_n) \left(\frac{L(k+1, g)}{(k+1)} \left(\frac{x}{l_n}\right)^{k+1} + O\left(\left(\frac{x}{l_n}\right)^{\alpha+k}\right) \right) \qquad \text{from (2.5)}$$

$$= \frac{L(k+1, g)}{(k+1)} x^{k+1} \left(\frac{\zeta(k) + c_{k+1}}{\zeta(k+1)} + O\left(\frac{1}{x^{k-1}}\right) \right) + O(x^{\alpha+k} \text{ or } x^2)$$

from (3.5), (3.2)

depending on whether $\alpha + k \gtrless 2$.

$$\sum_2 = F_k(x) = \frac{L(k+1, g)}{k+1} x^{k+1} + O(x^{\alpha+k}) \qquad \text{from (2.5).}$$

This proves the first part of Theorem 4.1.

Case 2 ($k=1$). From (4.1)

$$\sum_1 = \sum_{l_n \leq x} f_1(l_n)\Phi(x/l_n); \qquad \sum_2 = F_1(x).$$

Apply Lemmas 2.1 and 2.2 to $\sum_1$ with $g(l_n) = \phi(l_n)$, $h(l_n) = f_1(l_n)$ and $z = x_1 = x_2 = x^{1/2}$. Then

$$\sum_1 = \sum_{l_n \leqq z} \phi(l_n) F_1(x/l_n) + \sum_{l_n \leqq z} f_1(l_n) \Phi(x/l_n) - \Phi(z) F_1(z)$$
$$= \sum_{11} + \sum_{12} - \sum_{13} \qquad \text{(say)}.$$

Now

$$\sum_{11} = \sum_{l_n \leqq z} \phi(l_n) F_1(x/l_n)$$
$$= \sum_{l_n \leqq z} \phi(l_n) \left(\frac{L(2, g)}{2} \left(\frac{x}{l_n}\right)^2 + O\left(\frac{x}{l_n}\right)^{1+\alpha} \right) \qquad \text{from (2.5)}$$
$$= \frac{L(2, g)}{2} x^2 \left(\frac{1}{\zeta(2)} \left(\log x^{1/2} - \frac{\zeta^1(2)}{\zeta(2)} + C_1 \right) + O(x^{(\alpha-1)/2}) \right)$$
$$+ O(x^{1+\alpha} \cdot x^{(1-\alpha)/2}) \qquad \text{from (3.3) and (3.4)},$$

$$\sum_{12} = \sum_{l_n \leqq z} f_1(l_n) \Phi(x/l_n)$$
$$= \sum_{l_n \leqq z} f_1(l_n) \left(\frac{1}{2\zeta(2)} \left(\frac{x}{l_n}\right)^2 + O\left(\frac{x}{l_n}\right)^{1+\alpha} \right) \qquad \text{from (3.1)}$$
$$= \frac{x^2}{2\zeta(2)} (L(2, g)(\log x^{1/2} + C_1) + L^1(2, g) + O(x^{(\alpha-1)/2}))$$
$$+ O(x^{1+\alpha} \cdot x^{(1-\alpha)/2}) \qquad \text{from (2.7) and (2.8)},$$

$$\sum_{13} = \Phi(z) F_1(z) = \left(\frac{x}{2\zeta(2)} + O(x^{(1+\alpha)/2}) \right) \left(\frac{L(2, g)}{2} x + O(x^{(1+\alpha)/2}) \right)$$
$$\text{from (3.1) and (2.5)}$$
$$= \frac{L(2, g)}{4\zeta(2)} x^2 + O(x^{3/2+\alpha/2}).$$

$$\sum_2 = F_1(x) = \frac{L(2, g) x^2}{2} + O(x^{1+\alpha}) \qquad \text{from (2.5)}.$$

Then

$$\sum_{l_p, l_q \leqq x} f_1((l_p, l_q)) = 2 \left(\sum_{11} + \sum_{12} - \sum_{13} \right) - \sum_2$$

and this proves Case 2 of Theorem 4.1. Put $g(l_n) = \mu(l_n)$ in Theorem 4.1. Then $f_k((l_p, l_q)) = \phi_k((l_p, l_q)^k)$ for k an integer $\geqq 1$. We obtain

COROLLARY 4.1. $k \geqq 2$.

$$\sum_{l_p, l_q \leqq x} \phi_k((l_p, l_q)^k) = \frac{1}{(k+1)\zeta^2(k+1)} (2\zeta(k) + 2c_{k+1} - \zeta(k+1)) x^{k+1}$$
$$+ O(x^{\alpha+k});$$

$k = 1$.

$$\sum_{l_p, l_q \leq x} \phi((l_p, l_q))$$

$$= \frac{x^2}{\zeta^2(2)} \left\{ \log x + 2C_1 - \frac{1}{2} - \frac{\zeta(2)}{2} - \frac{2\zeta^1(2)}{\zeta(2)} \right\} + O(x^{(3+\alpha)/2}).$$

5. **When** $\alpha = 0$. Assumption (1.2) becomes

$$[x] = x + R(x) \quad \text{where} \quad R(x) = O(1). \tag{5.1}$$

Put $\alpha = 0$ in (2.21), (2.22), (2.24), and (2.28) and the results remain true. Using Abel's transformation the following sums may be found:

$$\sum_{l_n \leq x} \frac{\log l_n}{l_n} = \frac{1}{2} \log^2 x + \delta_1 + O\left(\frac{\log x}{x}\right), \tag{5.2}$$

$$\sum_{l_n \leq x} \frac{\log l_n}{l_n^\beta} = \frac{x^{1-\beta} \log x}{(1 - \beta)} - \frac{x^{1-\beta}}{(1 - \beta)^2} + \delta_\beta + O(x^{-\beta} \log x), \tag{5.3}$$

$$\sum_{l_n \leq x} \frac{R(l_n)}{l_n} = O(\log x). \tag{5.4}$$

Then (5.2) replaces (2.26), (5.4) replaces (2.28) and (5.1) replaces (2.25). The work is carried through as for $\alpha \neq 0$ and the results obtained are exactly the same as those in [2] except for different values of the constants.

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