DIFFERENTIAL OPERATORS WITH A PURELY CONTINUOUS SPECTRUM

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We consider a formally self-adjoint differential operator

$$L = -\frac{1}{r(x)}\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$$

whose coefficients are positive and of class C^1 on $[0, \infty)$. Let $\mathcal{L}_r^2(0, \infty)$ denote the class of complex valued functions $v(x)$ which satisfy

$$\|v\|^2 = \int_0^\infty r|v|^2 dx < \infty$$

and assume that L is in the limit point case at ∞. Then the boundary value problem

$$(1) \qquad \begin{aligned} Lu &= \lambda u, \\ u(0) &= 0; \qquad \|u\| < \infty \end{aligned}$$

gives rise to an operator $\bar{L}$ which is self-adjoint in $\mathcal{L}_r^2$.

The question which we shall investigate is: When is the spectrum of $\bar{L}$ purely continuous? One such result is given in [1, Chapter 9, problem 4] for the case $p = r \equiv 1$. This problem states that if

$$\int_0^\infty |q(x)| dx < \infty$$

then the spectral function $\rho(\lambda)$ corresponding to $\bar{L}$ is continuous (and in fact of class C^1) on $[0, \infty)$. A different result in this direction is given by the following[1]

THEOREM. *If $p' \leqq 0$, $r' \geqq 0$, and $(rq)' \leqq 0$ on $(0, \infty)$, then the spectrum of $\bar{L}$ is purely continuous.*

PROOF. Since the eigenfunction corresponding to any isolated eigenvalue of $\bar{L}$ belongs to $\mathcal{L}_2^r$, it is sufficient to show that for any $\lambda > 0$ every solution of

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[1] *Referee's comment.* The hypotheses of the author's theorem automatically ensure that L is in the limit point case at $x = +\infty$, so that it is unnecessary to make this an additional assumption.

$$
(1') \qquad \begin{aligned} Lu &= \lambda u, \\ u(0) &= 0 \end{aligned}
$$

fails to satisfy the condition $\lim_{t\to\infty} \int_0^t r|u|^2 dx < \infty$. If u is a solution of (1′) then we have

$$
(pu'' + p'u' + r(\lambda - q)u)u' = 0.
$$

This is equivalent to

$$
(pu'^2)' + p'u'^2 + (r(\lambda - q)u^2)' - (r(\lambda - q))'u^2 = 0.
$$

Integrating from 0 to t

$$
\int_0^t (-p'u'^2 + (r(\lambda - q))'u^2)dx + p(0)u'(0)^2 = p(t)u'(t)^2 + r(\lambda - q)u(t)^2.
$$

Let $k = p(0)u'(0)^2$. Since $p(0) > 0$ and $u'(0) \neq 0$, $k > 0$. Our hypotheses also guarantee that the above integral is a monotonic increasing function of t. Therefore we have

$$
(2) \qquad \int_0^T (pu'^2 + r(\lambda - q)u^2)dx \geqq kT.
$$

We shall refer to this inequality later.

If u is a solution of (1′) we also have

$$
0 = \int_0^t ((pu')' + r(\lambda - q)u)u dx,
$$

$$
0 = -\int_0^t pu'^2 dx + \int_0^t r(\lambda - q)u^2 dx + pu'u\Big]_0^t.
$$

Or finally

$$
(3) \qquad \int_0^t (pu'^2 + r(\lambda - q)u^2)dx = 2\int_0^t r(\lambda - q)u^2 dx + \frac{1}{2} p(t)(u(t)^2)'
$$

We consider two possibilities:

A. If there exists a positive constant M for which $(u(t)^2)' > 0$ for all $t > M$, then clearly u does not belong to $\mathfrak{L}_r^2$.

B. If there exists a sequence of positive numbers $\{t_n\}$ for which $t_n \uparrow \infty$ and $(u(t_n)^2)' \leqq 0$ then by (2) and (3) we conclude that

$$
2\int_0^{t_n} r(\lambda - q)u^2 dx \geqq \int_0^{t_n} (pu'^2 + r(\lambda - q)u^2)dx \geqq kt_n.
$$

Since rq is positive we have

$$\int_0^{t_n} ru^2 dx \geqq \frac{k}{2\lambda} t_n.$$

Again we conclude that $u \notin \mathfrak{L}_r^2$.

BIBLIOGRAPHY

1. E. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

2. F. Rellich, *Über das Asymptotische Verhalten der Lösungen von* $\Delta u + \lambda u = 0$ *in unendlichen Gebieten*, Jber. Deutsch. Math.-Verein. **53** (1943), 57–65.

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