BOUNDS FOR THE SOLUTIONS OF A CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Let $\Delta=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_n^2$ denote the n-dimensional Laplace operator and let the symbols D_r and S_r stand for the open sphere $x_1^2+\cdots+x_n^2<r^2$ $(r>0)$ and its boundary $x_1^2+\cdots+x_n^2=r^2$, respectively. We are concerned here with functions $u=u(P)$ $(P\in D_R)$ which are of class C^2 in D_R and satisfy there the differential equation

$$\Delta u = f(u), \tag{1}$$

or, more generally, the differential inequality

$$\Delta u \geqq f(u). \tag{2}$$

In the literature on the subject [1; 2; 3; 5; 7; 8], two closely related problems are investigated:

(a) What are the conditions to be imposed on the function $f(u)$ in order to guarantee the existence of a bound $\phi(r)=\phi(r, R; f)$ such that

$$\underset{P\in S_r}{u(P)} \leqq \phi(r, R; f) \tag{3}$$

if u satisfies (2) in a region D_R with $R>r$?

(b) Under what conditions on $f(u)$ will (2) have no solutions which are of class C^2 in the entire n-space?

Clearly, the nonexistence of such solutions is assured whenever it can be shown that $\phi(0, R; f)\to-\infty$ for $R\to\infty$.

The most general conditions on $f(u)$ for which the existence of such bounds for the solutions of (2) have been established are [3; 5]: $f(u)>0$, $f'(u)\geqq 0$ for $-\infty<u<\infty$,

$$\int_0^\infty \left[\int_0^u f(t)dt\right]^{-1/2} du < \infty. \tag{4}$$

In fact, if $f(u)>0$ and $f'(u)\geqq 0$, condition (4) is both necessary and sufficient. It was also shown in [3] and [5] that the problem $u(P) = \max$ is solved by a spherically symmetric solution $\phi(r)$ of (1) for which $\phi(r)\to\infty$ for $r\to R$, i.e., by a solution of the ordinary differential equation

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$$\phi''(r) + \frac{n-1}{r}\phi'(r) = f(\phi) \tag{5}$$

for which $\phi'(0)=0$ and $\phi(r)\to\infty$ for $r\to R$. In those cases in which this solution can be found explicitly it is thus possible to determine the exact upper bound (3). An example is the two-dimensional equation $\Delta u = e^u$, which has the well-known solution

$$u = 2\log\frac{\sqrt{8}R}{R^2 - r^2}\,.$$

It was pointed out by Osserman [5] that an upper bound for $u(P)$ is given by any spherically symmetric function v of class C^2 which satisfies the differential inequality

$$\Delta v \leqq f(v) \tag{6}$$

and tends to ∞ as $r\to R$. We shall here use this remark to find explicit upper bounds for certain classes of functions f. The following statement also gives a lower bound for max $u(P)$, which is obtained with the help of a suitable function satisfying the inequality (2).

THEOREM I. *Let $f(u)$ be a positive, nondecreasing, differentiable function in $(-\infty, \infty)$, for which*

$$\int_u^\infty \frac{dt}{f(t)} \qquad (u > -\infty)$$

exists, and for which

$$[f(u)]^{1+\lambda}\int_u^\infty \frac{dt}{f(t)} \tag{7}$$

is a nondecreasing function of u for some non-negative λ. If

$$\phi(r) = \sup_{P\in S_r} u(P), \tag{8}$$

where $u(P)$ ranges over all functions of class C^2 in D_r which satisfy (2), *then*

$$\frac{c(\lambda)(R^2 - r^2)^2}{R^2} \leqq \int_{\phi(r)}^\infty \frac{dt}{f(t)} \leqq \frac{R^2 - r^2}{2n}, \tag{9}$$

where

$$c(\lambda) = \frac{1}{4n} \qquad (4\lambda \leqq n-2) \tag{10}$$

and

$$c(\lambda) = \frac{1}{8(2\lambda + 1)} \qquad (4\lambda > n - 2). \tag{11}$$

The left-hand inequality (9) (*which yields the upper bound for* $\phi(r)$) *is sharp in the sense that for each number of dimensions* n ($n \geqq 2$), *there exists an equation* (1) *with a spherically symmetric solution* $\phi(r)$ *for which the sign of equality holds.*

The condition that (7) be a nondecreasing function of u is equivalent to the inequality

$$f'(u) \int_u^\infty \frac{dt}{f(t)} \leqq 1 + \lambda. \tag{12}$$

It is worth noting that this inequality is always satisfied, for $\lambda = 0$, if $\log f(u)$ is a convex function of u. Indeed, since f'/f is in this case a nondecreasing function of u, we have

$$\frac{1}{f(u)} = \int_u^\infty \frac{f'(t)}{f^2(t)} dt \geqq \frac{f'(u)}{f(u)} \int_u^\infty \frac{dt}{f(t)},$$

and the assertion follows. This implies the following special result.

If $\log f(u)$ *is a convex nondecreasing function in* $(-\infty, \infty)$, *and* $\phi(r)$ *is defined as before, then*

$$\frac{(R^2 - r^2)^2}{4nR^2} \leqq \int_{\phi(r)}^\infty \frac{dt}{f(t)} \leqq \frac{R^2 - r^2}{2n}. \tag{13}$$

In particular,

$$\frac{R^2}{4n} \leqq \int_{\phi(0)}^\infty \frac{dt}{f(t)} \leqq \frac{R^2}{2n}.$$

In the case of a solution u of

$$\Delta u = e^u \tag{14}$$

which is regular in D_R, (13) shows that

$$\log \frac{2n}{R^2 - r^2} \leqq \phi(r) \leqq 2 \log \frac{2\sqrt{n}R}{R^2 - r^2} \tag{15}$$

and, for $r = 0$,

$$2 \log \frac{\sqrt{(2n)}}{R} \leqq \phi(0) \leqq 2 \log \frac{2\sqrt{n}}{R}. \tag{16}$$

As already mentioned, the right-hand inequality (15) becomes an equality in the two-dimensional case. For $n\geqq 3$, no explicit solutions of (14) are known. However, it follows from the fact that the substitution of ρr for r and $u-2\log\rho$ for u $(\rho>0)$ transforms the equation into itself, that

$$\phi(0) = 2\log\frac{K_n}{R},$$

where K_n is a constant. (16) shows that $\sqrt{(2n)}\leqq K_n\leqq 2\sqrt{n}$. For $n=3$, an improved lower bound for $\phi(0)$ can be obtained from the observation that the 3-sphere of radius R is contained in the right circular cylinder of the same radius. Hence, $K_2<K_3$, and thus $2\sqrt{2}<K_3<2\sqrt{3}$.

2. Turning now to the proof of the left-hand inequality (9), we consider the function $v=v(r)$ defined by

$$\frac{c}{R^2}(R^2-r^2)^2=\int_v^\infty\frac{dt}{f(t)}\,. \tag{17}$$

We evidently have $v'(0)=0$, and $v(r)$ increases to ∞ as $r\to R$. If we can show that v satisfies the differential inequality (6), it will therefore follow that $\phi(r)\leqq v(r)$, and this will establish the left-hand inequality (9). To verify (6), we write x for any of the variables $x_1,\cdots,x_n$, and we differentiate (17) twice with respect to x. This yields

$$-\frac{4cx}{R^2}(R^2-r^2)=-\frac{v_x}{f(v)},$$

$$-\frac{4c}{R^2}(R^2-r^2)+\frac{8cx^2}{R^2}=-\frac{v_{xx}}{f(v)}+\frac{v_x^2f'(v)}{f^2(v)}$$

$$=-\frac{v_{xx}}{f(v)}+\frac{16c^2x^2}{R^4}(R^2-r^2)^2f'(v).$$

Summing over all the x_k, we obtain

$$-\frac{4cn}{R^2}(R^2-r^2)+\frac{8cr^2}{R^2}=-\frac{\Delta v}{f(v)}+\frac{16c^2r^2}{R^4}(R^2-r^2)^2f'(v),$$

or, in view of (17),

$$\frac{\Delta v}{f(v)}=\frac{16cr^2}{R^2}f'(v)\int_v^\infty\frac{dt}{f(t)}+\frac{4nc}{R^2}(R^2-r^2)-\frac{8cr^2}{R^2}\,. \tag{18}$$

Condition (12) therefore leads to the inequality

$$\frac{\Delta v}{f(v)} \leqq 4c\left[n - \frac{r^2}{R^2}(n - 2 - 4\lambda)\right]. \tag{19}$$

If $4\lambda \leqq n-2$, it follows that $\Delta v \leqq 4ncf(v)$, and v will satisfy (6) if c is given the value (10). If $4\lambda > n-2$, the maximum of the right-hand side of (19) (for $0 \leqq r \leqq R$) is attained for $r=R$, and the value (11) for c again leads to a function for which (6) holds.

The sign of equality in (9) will hold if v is a solution of $\Delta v = f(v)$. Since (19) was obtained from (18) by the use of the inequality (12), this is possible only if (12) becomes an equality. This will occur if

$$f(u) = u^{1+1/\lambda}, \qquad \lambda > 0,$$

and, for $\lambda = 0$, if

$$f(u) = e^u.$$

Furthermore, the right-hand side of (19) will be equal to the constant 1 only if the coefficient of r^2 vanishes (and, of course, if c is chosen in accordance with (10)). We thus must have $4\lambda = n-2$. Hence, the left-hand inequality (9) will become an equality in the case of the equation

$$\Delta u = u^{(n+2)/(n-2)}, \qquad n \geqq 3, \tag{20}$$

and, if $n=2$, the equation (14). The solution of (20) obtained in this way is easily confirmed to be of the form

$$u = \left[\frac{R\sqrt{(n(n-2))}}{R^2 - r^2}\right]^{8/(n-2)}.$$

This, incidentally, seems to be the only n-dimensional equation of the form $\Delta u = u^k$, $k > 1$, for which a solution can be obtained in terms of elementary functions.

It should be remarked here that, strictly speaking, the equation $\Delta u = u^k$ is not covered by Theorem I, since the conditions on $f(u)$ are satisfied only for $u > 0$. It is, however, clear that the left-hand inequality (9) will remain valid for solutions of this equation which are positive in D_R. It is also possible to give a more general version of Theorem I which applies to cases in which the hypotheses on $f(u)$ are satisfied only for $u > \alpha$, where α is a given number. Before we formulate this generalization, we prove the right-hand inequality (9).

The function $w = w_\rho(r)$ defined by

$$\frac{\rho^2 - r^2}{2n} = \int_w^\infty \frac{dt}{f(t)}, \qquad \rho > R,$$

is of class C^2 in D_R, and it satisfies the differential inequality (2). Indeed, differentiating with respect to $x = x_k$, we obtain

$$-\frac{x}{n} = -\frac{w_x}{f(w)},$$

$$-\frac{1}{n} = -\frac{w_{xx}}{f(w)} + \frac{w_x^2 f'(w)}{f^2(w)}$$

$$= -\frac{w_{xx}}{f(w)} + \frac{x^2}{n^2} f'(w),$$

and, summing over all the x_k,

$$\frac{\Delta w}{f(w)} = 1 + \frac{r^2}{n^2} f'(w).$$

Since $f'(w) \geqq 0$, it follows that $\Delta w \geqq f(w)$, and thus, in view of the results quoted above, $w(r) \leqq \phi(r)$. Since ρ may be taken arbitrarily close to R, this establishes the right-hand inequality (9). It may also be noted that the only assumption used was $f'(u) \geqq 0$; this estimate is therefore valid in the most general case in which the existence of $\phi(r)$ was proved in [3] and [5].

3. We now state the more general version of Theorem I.

THEOREM II. *Let $f(u)$ be continuous in $(-\infty, \infty)$, but satisfy the other hypotheses of Theorem* I *only for $u > \alpha$, where α is a given number; furthermore, let*

$$\int_\alpha^\infty \frac{dt}{f(t)} = \infty. \tag{21}$$

If u is a function of class C^2 in D_r satisfying the inequality (2), *and if $P \in S_r$ $(r < R)$, then either $u(P) \leqq \alpha$ or, if $u = u(P) > \alpha$,*

$$\frac{c(\lambda)}{R^2}(R^2 - r^2)^2 \leqq \int_u^\infty \frac{dt}{f(t)}. \tag{22}$$

The proof is again based on the fact that the function v defined in (17) tends to ∞ for $r \rightarrow R$ and satisfies the inequality (6). Condition

(21) guarantees that this definition is meaningful for all given values of R and all r in $[0, R)$. The fact that $u \leqq v$, if u satisfies (2), now follows by a slight modification of the argument used in [5]. Since $\Delta v \leqq f(v)$, we have

$$\Delta(u - v) \geqq f(u) - f(v). \tag{23}$$

Suppose there exists a nonempty set T in D_R on which $u > v$. Since $v > \alpha$, we necessarily have $u > \alpha$ on T, and it follows from our assumptions that $f(u) \geqq f(v)$ on this set. By (23), $u - v$ is therefore subharmonic on T. We may assume that u is of class C^2 on $D_R + S_R$; this assumption can then be removed by a standard argument. Since $v \to \infty$ for $r \to R$, we have $u - v \to -\infty$ for $r \to R$, and it follows that the boundary B of T is in D_R. Hence, $u - v = 0$ on B, which is absurd, since $u - v$ is positive and subharmonic on T. The set T is thus empty, and we must have $u \leqq v$ throughout D_R. This proves (22).

An immediate consequence of Theorem II is the following result concerning the nonexistence of certain types of entire solutions.

THEOREM III. *If $f(u)$ is subject to the hypotheses of Theorem* II, *and if there exists a function u satisfying* (2) *which is of class C^2 in the entire space, then*

$$u(P) \leqq \alpha \tag{24}$$

for all P.

Indeed, suppose that $u(P) > \alpha$ for some point P. Since (2) remains unchanged under a translation of the coordinate system, we may take P to be the origin. It follows therefore from (22) that

$$c(\lambda) R^2 \leqq \int_{u(P)}^{\infty} \frac{dt}{f(t)},$$

and this produces a contradiction if R is taken large enough.

That entire solutions satisfying (24) may indeed exist is shown by the equation $\Delta u = u^{k+1}$, where k is a positive integer, which satisfies all the assumptions for $\alpha = 0$, and which has the trivial entire solution $u \equiv 0$. A nontrivial example is given by the equation

$$\Delta u = u^2 + u$$

in three dimensions, for which $\alpha = 0$, and which is known to possess a negative entire solution [4; 6].

As a final example, we mention the equation $\Delta u = u^{2k+1}$, where k is a positive integer. Since the equation remains unchanged if u is replaced by $-u$, it follows from Theorem III that, except for the trivial solution $u \equiv 0$, the equation has no entire solutions.

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