If $X$ is a set, let $I^X$ be all functions from $X$ into the unit interval $I$. Note that if $f$ and $g$ are in $I^X$ then so are $1-f$ and $fg$. Such a collection of functions is said to have property V. That is, $F$ has property V in case

(i) $F \subseteq I^X$ for some set $X$,
(ii) $f \in F$ implies $1-f \in F$,
(iii) $f, g \in F$ implies $fg \in F$.

Giving $I^X$ the topology of uniform convergence, we have that the closure of a set with property V has property V, as does the intersection of such sets. Thus every subset of $I^X$ is contained in a smallest set with property V, and in a smallest closed set with property V. If $X$ is a topological space then the set $D(X)$ of all continuous functions from $X$ into $I$ is closed and has property V. The idea of considering such collections of functions comes from a statement of von Neumann in [1]. Essentially, he claims without proof what we give here as a corollary to Theorem 2. I am indebted to Dr. R. S. Pierce for bringing the problem to my attention.

**Definition.** If $n$ is a positive integer, let $P_n$ be the smallest subset of $D(I^n)$ that has property V and contains the $n$ projections.

**Lemma 1.** Let $F$ have property V, $p \in P_n$, and $f_k \in F$ for $k = 1, 2, \ldots, n$. Then the function $f$ defined by

$$f(x) = p(f_1(x), f_2(x), \ldots, f_n(x))$$

is in $F$.

**Proof.** Let $Q$ be the set of all $q \in D(I^n)$ for which $q(f_1(x), f_2(x), \ldots, f_n(x))$ is in $F$. Then $Q$ has property V and contains the $n$ projections. So $Q$ contains $P_n$.

**Lemma 2.** If $a < b$ and $\epsilon > 0$, then there exists $p \in P_1$ such that

- $p > 1 - \epsilon$ in $[0, a]$,
- $p < \epsilon$ in $[b, 1]$.

We set $[0, a] = \emptyset$ if $a < 0$ and $[b, 1] = \emptyset$ if $b > 1$.

**Proof.** Since for a sufficiently large integer $k$, $x^k(1-x)^k < \epsilon$ and

Received by the editors August 2, 1962.
1 - x^k(1 - x)^k > 1 - \epsilon for all x in I, we can assume that 0 \leq a and b \leq 1. Also, since there exist a', b' such that a < a' < b' < b, we can assume that 0 < a < b < 1. Now our solution will be of the form \( p(x) = (1 - x^m)^n \).

Pick \( r \) such that \( \left( \frac{3}{4} \right)^r < \epsilon \). Pick \( m, s \) such that

\[
\left( \frac{3}{4} \right)^s < s < \frac{1}{b^m} < \left( \frac{\epsilon}{r} \right)^{1/a^m}.
\]

Let \( n = rs \) and note that \( na^m < \epsilon, \frac{3}{4} < sb^m < 1 \). So

\[
(1 - a^m)^n > 1 - na^m > 1 - \epsilon,
\]

\[
(1 - b^m)^n = [(1 - b^m)^s]^r < \left[ 1 - sb^m + \frac{1}{2} (sb^m)^2 \right]^r < \left( \frac{3}{4} \right)^r < \epsilon.
\]

One can prove by induction that if 0 < x < 1 then (1 - x)^n < 1 - nx + \frac{1}{2}(nx)^2.

**Lemma 3.** If \( a_k, b_k \in I \) for \( k = 1, 2, \cdots, n \) then

\[
\left| \prod_{1}^{n} a_k - \prod_{1}^{n} b_k \right| \leq \sum_{1}^{n} | a_k - b_k |.
\]

**Proof.** The induction step can be verified as follows. Let \( a = a_1a_2\cdots a_{n-1} \) and \( b = b_1b_2\cdots b_{n-1} \). So \( a, b \in I \) and

\[
| aa_n - bb_n | \leq | aa_n - ba_n | + | ba_n - bb_n | \leq | a - b | + | a_n - b_n |.
\]

**Lemma 4.** Let \( (a, b) \in I \times I \) and \( \epsilon, \delta > 0 \). Then there exists \( p \in P_3 \) such that

\[
p(x, y) > 1 - \epsilon \text{ if } (x - a)^2 + (y - b)^2 \leq \delta^2,
p(x, y) < \epsilon \text{ if } (x - a)^2 + (y - b)^2 \geq (4\delta)^2.
\]

**Proof.** Let the functions \( p_1, p_2, p_3, p_4 \in P_1 \) correspond by Lemma 2 to \( a - 2\delta < a - \delta, a + \delta < a + 2\delta, b - 2\delta < b - \delta, b + \delta < b + 2\delta \) and \( \epsilon/4 > 0 \), respectively. Then let \( \rho \) be given by

\[
\rho(x, y) = [1 - p_1(x)]p_2(x)[1 - p_3(y)]p_4(y).
\]

**Lemma 5.** Let \( A, B \in I \times I \) be closed and disjoint. If \( \epsilon > 0 \) and \( p \in P_3 \), then there exists \( q \in P_3 \) such that

\[
q \geq p \text{ in } I \times I,
q > 1 - \epsilon \text{ in } A,
q < p + \epsilon \text{ in } B.
\]
Proof. We can assume that $A$ and $B$ are nonvoid. Let $4\delta = \text{dist}(A, B)$. Then $\delta > 0$ and there exist $(c_k, d_k) \in A$ for $k = 1, 2, \ldots, n$ such that the $\delta$-neighborhoods of the $(c_k, d_k)$ cover $A$. For each $k$ there exists $g_k \in P_2$ such that

\begin{align*}
q_k(x, y) > 1 - \frac{\epsilon}{n} & \text{ if } (x - c_k)^2 + (y - d_k)^2 \leq \delta^2, \\
q_k(x, y) < \frac{\epsilon}{n} & \text{ if } (x - c_k)^2 + (y - d_k)^2 \geq (4\delta)^2.
\end{align*}

Let $q_0 = (1 - q_1)(1 - q_2) \cdots (1 - q_n)$. It is clear that $q_0 > 1 - \epsilon$ in $B$, and $q_0 < \epsilon/n$ in $A$. Now let $q = 1 - (1 - p)q_0$. In $I \times I$ we have $q \geq 1 - (1 - p) = p$. In $A$ we have $q \geq 1 - q_0 > 1 - \epsilon$. And in $B$ we have $q - p = 1 - q_0 + pq_0 - p = (1 - q_0)(1 - p) < \epsilon$.

Theorem 1. Let $X$ be a set and $F$ a closed subset of $I^X$. If $F$ has property $V$ then $F$ is a lattice.

Proof. In view of Lemma 1, it is enough to prove that the functions $(x, y) \rightarrow x \wedge y$ and $(x, y) \rightarrow x \vee y$ of $I \times I$ into $I$ can be uniformly approximated by members of $P_2$. Since $x \vee y = 1 - (1 - x) \wedge (1 - y)$, it is enough to check $x \wedge y$. Let $0 < \epsilon < \frac{1}{4}$ and let $C$ be the set of all $(x, y) \in I \times I$ for which $\epsilon \leq x \wedge y \leq 1 - \epsilon$. Then $C$ is closed and there exists $m > 0$ such that $x^m y^m < \epsilon$ in $C$. Let $p(x, y) = 1 - x^m y^m$. Then $1 - \epsilon < p < 1$ in $C$. For $k \geq 0$ let

\begin{align*}
A_k &= \{ (x, y) \in C \mid p^k(x, y) \geq x \wedge y \}, \\
B_k &= \{ (x, y) \in C \mid p^k(x, y) \leq x \wedge y \}.
\end{align*}

Then $A_1 = C$ and for $k \geq 0$

\begin{align*}
A_k &\supset A_{k+1}, \\
B_k &\supset C - A_k, \\
A_{k+1} \cap B_k &= \emptyset.
\end{align*}

Because the $A_k$ have void intersection, there exists $n > 2$ such that $A_n = \emptyset$. For $k = 1, 2, \ldots, n$, pick $q_k \in P_2$ such that $q_k \geq p$ in $I \times I$, $q_k > 1 - \epsilon/n$ in $B_{k-1}$, and $q_k < p + \epsilon/n$ in $A_k$. Let $q = q_1 q_2 \cdots q_n$. Now $C = \bigcup_{k=1}^{n-1} (A_k - A_{k+1})$. For $k = 1, 2, \ldots, n - 1$ we have in $A_k - A_{k+1}$

\begin{align*}
0 \leq p^k - x \wedge y < p^k - p^{k+1} = p^k(1 - p) < \epsilon.
\end{align*}

Also, we have

\begin{align*}
| p^k - q | &\leq | p^k - p^{k+1} | + | p^{k+1} - q | < \epsilon + \left| p^{k+1} - \prod_{i=1}^{n} q_j \right| \\
&\leq \epsilon + \sum_{i=1}^{k} | p - q_j | + | p - q_{k+1} | + \sum_{j=k+2}^{n} | 1 - q_j | \\
&< \epsilon + k \frac{\epsilon}{n} + (1 - p) + (n - k - 1) \frac{\epsilon}{n} < 3\epsilon.
\end{align*}
Thus in $C$, $|q - x \wedge y| < 4\varepsilon$. Now by Lemma 5, there exists $q' \in \mathbb{P}_1$ such that $q' \geq q$ in $I \times I$, $q' > 1 - \varepsilon$ if $x \wedge y \geq 1 - \varepsilon$, and $q' < q + \varepsilon$ if $x \wedge y \leq 1 - 2\varepsilon$. Clearly $|q' - x \wedge y| < 6\varepsilon$ if $x \wedge y \geq \varepsilon$. Similarly, there exists $q'' \in \mathbb{P}_1$ such that $q'' \leq q'$ in $I \times I$, $q'' < \varepsilon$ if $x \wedge y \leq \varepsilon$, and $q'' > q' - \varepsilon$ if $x \wedge y \geq 2\varepsilon$. So $|q'' - x \wedge y| < 8\varepsilon$ in all of $I \times I$.

**Theorem 2.** Let $X$ be a compact space and $F$ a closed, point-separating subset of $D(X)$ that has property V. If $S$ is the set of points of $X$ taken into the doubleton $\{0, 1\}$ by every member of $F$, then $F$ consists of all functions $f \in D(X)$ for which $f(S) \subseteq \{0, 1\}$.

**Proof.** It is well known [2] that for a compact space $Y$, a closed sublattice of $C(Y)$ contains any continuous function which it approximates at each pair of points. So, let $f \in D(X)$ be such that $f(S) \subseteq \{0, 1\}$, and let $u, v$ be distinct elements of $X$. The case when $X$ has only one element is straightforward.

Suppose $u, v \in S$. Then there exists $g \in F$ such that $g(u) \neq g(v)$, and then one of $g, 1 - g, 1 - g(1 - g)$ duplicates $f$ on $u$ and $v$.

If $u \in S, v \in S$, then there exists $g \in F$ such that $g(u) = f(u)$ and $g(v) \in (0, 1)$. As can be seen from Lemmas 1 and 2, something of the form $1 - (1 - g^n)$ will do.

If $u, v \notin S$, then there exist $g_1, g_2, g_3 \in F$ such that $g_1(v) \leq g_1(u) \in (0, 1)$, $g_2(u) \leq g_2(v) \in (0, 1)$, and $g_3(u) = g_3(v)$. If we let $h_1 = g_1g_2$ and $h_2 = g_3(1 - g_3)$, we have $h_1(v) < h_1(u) < 1$ and $h_2(u) < h_2(v) < 1$. Something of the form $f_2 = (1 - h_2^m)$ approximates $1$ at $u$ and $f$ at $v$. Something of the form $f_1 = (1 - h_1^m)$ approximates $f$ at $u$ and $1$ at $v$. So $f_1f_2$ approximates $f$ at $u$ and $v$.

**Corollary.** The smallest closed subset of $D(I^\infty)$ having property V and containing the projections and at least one constant $c \in (0, 1)$ is $D(I^\infty)$ itself.

**References**


University of Oregon