A VARIATION ON THE STONE-WEIERSTRASS THEOREM

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If X is a set, let I^X be all functions from X into the unit interval I. Note that if f and g are in I^X then so are $1-f$ and fg. Such a collection of functions is said to have property V. That is, F has property V in case

(i) $F \subset I^X$ for some set X,
(ii) $f \in F$ implies $1-f \in F$,
(iii) $f, g \in F$ implies $fg \in F$.

Giving I^X the topology of uniform convergence, we have that the closure of a set with property V has property V, as does the intersection of such sets. Thus every subset of I^X is contained in a smallest set with property V, and in a smallest closed set with property V. If X is a topological space then the set $D(X)$ of all continuous functions from X into I is closed and has property V. The idea of considering such collections of functions comes from a statement of von Neumann in [1]. Essentially, he claims without proof what we give here as a corollary to Theorem 2. I am indebted to Dr. R. S. Pierce for bringing the problem to my attention.

Definition. If n is a positive integer, let P_n be the smallest subset of $D(I^n)$ that has property V and contains the n projections.

Lemma 1. *Let F have property* V, *$p \in P_n$, and $f_k \in F$ for $k=1, 2, \cdots, n$. Then the function f defined by*

$$f(x) = p(f_1(x), f_2(x), \cdots, f_n(x))$$

is in F.

Proof. Let Q be the set of all $q \in D(I^n)$ for which $q(f_1(x), f_2(x), \cdots, f_n(x))$ is in F. Then Q has property V and contains the n projections. So Q contains P_n.

Lemma 2. *If $a<b$ and $\epsilon>0$, then there exists $p \in P_1$ such that*

$$p > 1 - \epsilon \text{ in } [0, a],$$
$$p < \epsilon \quad \text{ in } [b, 1].$$

We set $[0, a]=\varnothing$ if $a<0$ and $[b, 1]=\varnothing$ if $b>1$.

Proof. Since for a sufficiently large integer k, $x^k(1-x)^k<\epsilon$ and

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$1-x^k(1-x)^k>1-\epsilon$ for all x in I, we can assume that $0\leqq a$ and $b\leqq 1$. Also, since there exist a', b' such that $a<a'<b'<b$, we can assume that $0<a<b<1$. Now our solution will be of the form $p(x)=(1-x^m)^n$. Pick r such that $(\frac{3}{4})^r<\epsilon$. Pick m, s such that

$$\left(\frac{3}{4}\right)\frac{1}{b^m}<s<\frac{1}{b^m}<\left(\frac{\epsilon}{r}\right)\frac{1}{a^m}.$$

Let $n=rs$ and note that $na^m<\epsilon$, $\frac{3}{4}<sb^m<1$. So

$$(1-a^m)^n>1-na^m>1-\epsilon,$$

$$(1-b^m)^n=[(1-b^m)^s]^r<\left[1-sb^m+\frac{1}{2}(sb^m)^2\right]^r<\left(\frac{3}{4}\right)^r<\epsilon.$$

One can prove by induction that if $0<x<1$ then $(1-x)^n<1-nx+\frac{1}{2}(nx)^2$.

LEMMA 3. *If a_k, $b_k\in I$ for $k=1, 2, \cdots, n$ then*

$$\left|\prod_1^n a_k-\prod_1^n b_k\right|\leqq\sum_1^n|a_k-b_k|.$$

PROOF. The induction step can be verified as follows. Let $a=a_1a_2\cdots a_{n-1}$ and $b=b_1b_2\cdots b_{n-1}$. So a, $b\in I$ and

$$\begin{aligned}|aa_n-bb_n|&\leqq|aa_n-ba_n|+|ba_n-bb_n|\\&\leqq|a-b|+|a_n-b_n|.\end{aligned}$$

LEMMA 4. *Let $(a, b)\in I\times I$ and ϵ, $\delta>0$. Then there exists $p\in P_2$ such that*

$$p(x,y)>1-\epsilon \text{ if } (x-a)^2+(y-b)^2\leqq\delta^2,$$
$$p(x,y)<\epsilon \text{ if } (x-a)^2+(y-b)^2\geqq(4\delta)^2.$$

PROOF. Let the functions p_1, p_2, p_3, $p_4\in P_1$ correspond by Lemma 2 to $a-2\delta<a-\delta$, $a+\delta<a+2\delta$, $b-2\delta<b-\delta$, $b+\delta<b+2\delta$ and $\epsilon/4>0$, respectively. Then let p be given by

$$p(x,y)=[1-p_1(x)]p_2(x)[1-p_3(y)]p_4(y).$$

LEMMA 5. *Let A, $B\subset I\times I$ be closed and disjoint. If $\epsilon>0$ and $p\in P_2$, then there exists $q\in P_2$ such that*

$$q\geqq p \text{ in } I\times I,$$
$$q>1-\epsilon \text{ in } A,$$
$$q<p+\epsilon \text{ in } B.$$

PROOF. We can assume that A and B are nonvoid. Let $4\delta = \text{dist}(A, B)$. Then $\delta > 0$ and there exist $(c_k, d_k) \in A$ for $k = 1, 2, \cdots, n$ such that the δ-neighborhoods of the (c_k, d_k) cover A. For each k there exists $q_k \in P_2$ such that

$$q_k(x, y) > 1 - \epsilon/n \text{ if } (x - c_k)^2 + (y - d_k)^2 \leqq \delta^2,$$
$$q_k(x, y) < \epsilon/n \text{ if } (x - c_k)^2 + (y - d_k)^2 \geqq (4\delta)^2.$$

Let $q_0 = (1 - q_1)(1 - q_2) \cdots (1 - q_n)$. It is clear that $q_0 > 1 - \epsilon$ in B, and $q_0 < \epsilon/n$ in A. Now let $q = 1 - (1 - p)q_0$. In $I \times I$ we have $q \geqq 1 - (1 - p) = p$. In A we have $q \geqq 1 - q_0 > 1 - \epsilon$. And in B we have $q - p = 1 - q_0 + pq_0 - p = (1 - q_0)(1 - p) < \epsilon$.

THEOREM 1. *Let X be a set and F a closed subset of I^X. If F has property* V *then F is a lattice.*

PROOF. In view of Lemma 1, it is enough to prove that the functions $(x, y) \rightarrow x \wedge y$ and $(x, y) \rightarrow x \vee y$ of $I \times I$ into I can be uniformly approximated by members of P_2. Since $x \vee y = 1 - (1 - x) \wedge (1 - y)$, it is enough to check $x \wedge y$. Let $0 < \epsilon < \frac{1}{4}$ and let C be the set of all $(x, y) \in I \times I$ for which $\epsilon \leqq x \wedge y \leqq 1 - \epsilon$. Then C is closed and there exists $m > 0$ such that $x^m y^m < \epsilon$ in C. Let $p(x, y) = 1 - x^m y^m$. Then $1 - \epsilon < p < 1$ in C. For $k \geqq 0$ let

$$A_k = \{(x, y) \in C \mid p^k(x, y) \geqq x \wedge y\},$$
$$B_k = \{(x, y) \in C \mid p^k(x, y) \leqq x \wedge y\}.$$

Then $A_1 = C$ and for $k \geqq 0$

$$A_k \supset A_{k+1}, \qquad B_k \supset C - A_k, \qquad A_{k+1} \cap B_k = \varnothing.$$

Because the A_k have void intersection, there exists $n > 2$ such that $A_n = \varnothing$. For $k = 1, 2, \cdots, n$ pick $q_k \in P_2$ such that $q_k \geqq p$ in $I \times I$, $q_k > 1 - \epsilon/n$ in B_{k-1}, and $q_k < p + \epsilon/n$ in A_k. Let $q = q_1 q_2 \cdots q_n$. Now $C = \bigcup_1^{n-1} (A_k - A_{k+1})$. For $k = 1, 2, \cdots, n-1$ we have in $A_k - A_{k+1}$

$$0 \leqq p^k - x \wedge y < p^k - p^{k+1} = p^k(1 - p) < \epsilon.$$

Also, we have

$$\begin{aligned} |p^k - q| &\leqq |p^k - p^{k+1}| + |p^{k+1} - q| < \epsilon + \left| p^{k+1} - \prod_1^n q_j \right| \\ &\leqq \epsilon + \sum_1^k |p - q_j| + |p - q_{k+1}| + \sum_{k+2}^n |1 - q_j| \\ &< \epsilon + k\frac{\epsilon}{n} + (1 - p) + (n - k - 1)\frac{\epsilon}{n} < 3\epsilon. \end{aligned}$$

Thus in C, $|q - x \wedge y| < 4\epsilon$. Now by Lemma 5, there exists $q' \in P_2$ such that $q' \geqq q$ in $I \times I$, $q' > 1 - \epsilon$ if $x \wedge y \geqq 1 - \epsilon$, and $q' < q + \epsilon$ if $x \wedge y \leqq 1 - 2\epsilon$. Clearly $|q' - x \wedge y| < 6\epsilon$ if $x \wedge y \geqq \epsilon$. Similarly, there exists $q'' \in P_2$ such that $q'' \leqq q'$ in $I \times I$, $q'' < \epsilon$ if $x \wedge y \leqq \epsilon$, and $q'' > q' - \epsilon$ if $x \wedge y \geqq 2\epsilon$. So $|q'' - x \wedge y| < 8\epsilon$ in all of $I \times I$.

THEOREM 2. *Let X be a compact space and F a closed, point-separating subset of $D(X)$ that has property* V. *If S is the set of points of X taken into the doubleton $\{0, 1\}$ by every member of F, then F consists of all functions $f \in D(X)$ for which $f(S) \subset \{0, 1\}$.*

PROOF. It is well known [2] that for a compact space Y, a closed sublattice of $C(Y)$ contains any continuous function which it approximates at each pair of points. So, let $f \in D(X)$ be such that $f(S) \subset \{0, 1\}$, and let u, v be distinct elements of X. The case when X has only one element is straightforward.

Suppose $u, v \in S$. Then there exists $g \in F$ such that $g(u) \neq g(v)$, and then one of g, $1 - g$, $g(1 - g)$, $1 - g(1 - g)$ duplicates f on u and v.

If $u \in S$, $v \notin S$, then there exists $g \in F$ such that $g(u) = f(u)$ and $g(v) \in (0, 1)$. As can be seen from Lemmas 1 and 2, something of the form $1 - (1 - g^m)^n$ will do.

If $u, v \notin S$, then there exist $g_1, g_2, g_3 \in F$ such that $g_1(v) \leqq g_1(u) \in (0, 1)$, $g_2(u) \leqq g_2(v) \in (0, 1)$, and $g_3(v) < g_3(u)$. If we let $h_1 = g_1 g_3$ and $h_2 = g_2(1 - g_3)$, we have $h_1(v) < h_1(u) < 1$ and $h_2(u) < h_2(v) < 1$. Something of the form $f_2 = (1 - h_2^r)^s$ approximates 1 at u and f at v. Something of the form $f_1 = (1 - h_1^k)^m$ approximates f at u and 1 at v. So $f_1 f_2$ approximates f at u and v.

COROLLARY. *The smallest closed subset of $D(I^n)$ having property* V *and containing the projections and at least one constant $c \in (0, 1)$ is $D(I^n)$ itself.*

REFERENCES

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