A NOTE ON THE REPRESENTATION OF α-COMPLETE BOOLEAN ALGEBRAS

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It is a fundamental theorem of representation theory for Boolean algebras that every $\aleph_0$-complete Boolean algebra is an $\aleph_0$-homomorphic image of an $\aleph_0$-field of sets. It is also well known that there is a $2^{\aleph_0}$-complete Boolean algebra which is not a $2^{\aleph_0}$-homomorphic image of a $2^{\aleph_0}$-field of sets. The usual proof goes by constructing a complete Boolean algebra that is not $(\aleph_0, 2)$-distributive; that is, one that does not satisfy the equation

$$\prod_{\mu<\omega} \sum_{\nu<2} b_{\mu\nu} = \sum_{f\in 2^\omega} \prod_{\mu<\omega} b_{\mu f(\mu)}.$$

Since this equation involves only $2^{\aleph_0}$-operations and holds in $\aleph_0$-fields of sets, it also has to hold in $2^{\aleph_0}$-homomorphic images of $2^{\aleph_0}$-fields of sets. It was, however, an open question whether or not one could prove the existence of an $\aleph_1$-complete Boolean algebra not an $\aleph_1$-homomorphic image of an $\aleph_1$-field of sets without using the continuum hypothesis. This question is answered in this note. We construct a complete Boolean algebra which does not satisfy the inequality

$$\prod_{\nu<\omega_1} \sum_{\mu<\omega} b_{\nu\mu} \leqq \sum_{\nu\neq\nu'<\omega_1} \sum_{\mu<\omega} b_{\nu\mu}\cdot b_{\nu'\mu}. \tag{1}$$

Since this inequality involves only $\aleph_1$-operations and holds in $\aleph_1$-fields of sets, it also has to hold in $\aleph_1$-homomorphic images of $\aleph_1$-fields of sets.

From now on, let us identify a given cardinal $\aleph$ with the first ordinal number having cardinal $\aleph$, and identify a given ordinal number with its set of predecessors. If α is any cardinal number, let α^+ be the first cardinal larger than α. It is customary to call an α-complete Boolean algebra *α-representable* if it is an α-homomorphic image of an α-field of sets. Consider this question: Which cardinals α have the property

R_α: There is an α^+-complete α-representable Boolean algebra which is not α^+-representable?

It is known that regular infinite cardinals α have property R_α if $\alpha^+ = 2^\alpha$. Examples of complete α-representable algebras which are not $(\alpha, 2)$-distributive are given in Smith [6] and Scott [4]. In this note

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we show that all regular infinite cardinals α have property R_α making no use of any form of the continuum hypothesis. The proof goes by constructing a complete α-representable Boolean algebra that does not satisfy the inequality

$$(1)_\alpha \qquad \prod_{\nu<\alpha^+} \sum_{\mu<\alpha} b_{\nu\mu} \leqq \sum_{\nu\neq\nu'<\alpha^+} \sum_{\mu<\alpha} b_{\nu\mu}\cdot b_{\nu'\mu}.$$

These algebras are also (β, γ)-distributive for all cardinals $\beta<\alpha$ and all γ. They are not (α, α)-distributive.

The problem of determining which, if any, singular infinite cardinals have property R_α seems to be open, even assuming the generalized continuum hypothesis.

Let α be a regular infinite cardinal. Considering the set X of all one-to-one functions on α into α^+ as points, take as a basis for open sets the empty set, together with sets $A_g=\{f:f\in X \text{ and } f|\,\text{Dom } g=g\}$, where g is a one-to-one function on a subset of α having cardinal less than α, into α^+.

If $\{A_{g(i)}: i\in I\}$ is a collection of fewer than α nonempty basic sets, then one sees that $\bigcap_{i\in I} A_{g(i)}\neq\emptyset$ if and only if $\bigcup_{i\in I} g(i)$ is a one-to-one function. Since the regularity of α guarantees card $\bigcup_{i\in I}\text{Dom}(g(i))<\alpha$, $\bigcap_{i\in I} A_{g(i)}$ is either empty or is equal to A_g, where $g=\bigcup_{i\in I} g(i)$. Thus the collection of basic open sets is closed under intersections of fewer than α elements. Moreover, since $\bigcup_{i\in I} g(i)$ is a one-to-one function if and only if $g(i)\cup g(i')$ is a one-to-one function for each pair $i, i'\in I$, we have the following compactness property:

(*) If $\{A_{g(i)}: i\in I\}$ is a collection of fewer than α nonempty basic open sets such that no pair has an empty intersection, then $\bigcap_{i\in I} A_{g(i)}$ is a nonempty basic open set.

Basic sets are open-closed, since $X\sim A_g=X\sim\bigcap\{A_{\{(\mu\nu)\}}: (\mu\nu)\in g\}=\bigcup\{X\sim A_{\{(\mu\nu)\}}: (\mu\nu)\in g\}$, while for any pair $(\mu\nu)\in\alpha\times\alpha^+$,

$$X\sim A_{\{(\mu\nu)\}}=\bigcup\{A_{\{(\mu\nu')\}}: \nu\neq\nu'<\alpha^+\}.$$

Let B_α be the algebra of regular open sets of this space. This algebra consists of sets S such that $S=\text{in cl } S$ under operations

$$-S = \text{in}\,(X\sim S)$$

$$\sum_\xi S_\xi = \text{in cl} \bigcup_\xi S_\xi$$

$$\prod_\xi S_\xi = \text{in cl} \bigcap_\xi S_\xi.$$

Such algebras are always complete. See Sikorski's book [5] for details.

THEOREM. *Algebras B_α are α-representable and (β, γ)-distributive for all $\beta<\alpha$ and all cardinals γ. The inequality $(1)_\alpha$ does not hold in B_α. Hence B_α is not α^+-representable.*

PROOF. Property (*) implies that B_α is β-atomic for all $\beta<\alpha$. Therefore, for the distributivity of B_α, we can refer the reader to Pierce [3], where, in turn, he will be referred to [2]. The method in [3] for showing that β-atomicity implies β^+-representability, can also be used to show that β-atomicity for all $\beta<\alpha$ implies α-representability. One can conveniently use either the condition of Chang in [1] or of Smith in [6].

We claim that $\mathrm{cl} \bigcup_{\mu<\alpha} A_{\{(\mu\nu)\}}=X$ for any $\nu<\alpha^+$. For if A_g is any nonempty basic open set with $\nu\in\mathrm{Rng}(g)$, then $A_g\subseteq A_{\{(\mu\nu)\}}$ where $\mu=g^{-1}(\nu)$. If A_g is a nonempty basic open set with $\nu\notin\mathrm{Rng}(g)$, then we can choose $\mu\in\alpha\sim\mathrm{Dom}(g)$ since $\mathrm{Dom}(g)$ has cardinal less than α. For such a μ, $g\cup\{(\mu\nu)\}$ is a one-to-one function, and therefore $A_g\cap A_{\{(\mu\nu)\}}\neq\emptyset$.

In B_α, therefore, $\prod_{\nu<\alpha^+}\sum_{\mu<\alpha} A_{\{(\mu\nu)\}}=X$, the unit of the algebra. On the other hand, $A_{\{(\mu\nu)\}}\cdot A_{\{(\mu\nu')\}}=\emptyset$ for any $\nu\neq\nu'<\alpha^+$ and $\mu<\alpha$. Hence $(1)_\alpha$ fails in B_α.

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