ON COMMUTATORS IN A SIMPLE LIE ALGEBRA

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1. Introduction. K. Shoda [3] has shown that any \( n \times n \) matrix of trace zero over a field of characteristic zero is expressible as an additive commutator. A. Albert and B. Muckenhoupt [1] have extended this result to fields of all characteristics. Rephrasing their result, it is easily seen that every element of a Lie algebra of type \( A_n \) can be expressed as a Lie product. It seems natural to ask whether a similar assertion is valid for a wider class of Lie algebras.

The purpose of this paper is to employ a generalization of Shoda’s method to show that any element of a classical Lie algebra (defined by R. Steinberg [4]) can be expressed as a Lie product provided only that a certain restrictive assumption on the cardinality of the field \( K \) over which the algebra is defined is satisfied. This restriction is made in this paper for the sole purpose of permitting a relatively simple method to be applied to prove the main theorem.

2. The notation and method of proof. The classical simple Lie algebras will be denoted by \( L = L^*/Z \) where \( Z \) is the center of \( L^* \). \( L^* \) will also be called classical. \( K \) will denote the base field. \( xy \) is the Lie product of \( x \) and \( y \), \( H \) a standard Cartan subalgebra, \( \phi \) runs through the nonzero roots of \( L^* \) relative to \( H \), \( L^* = H + E \), a vector space direct sum, where \( E = \sum \phi L_\phi \). Each \( L_\phi \) is one-dimensional, and \( H \) is spanned by \( h_i = h_{\phi_i} \) (\( i = 1, \ldots, n \)), where \( \phi_i(h_i) = 2, h_i = e_{\phi_i} e_{-\phi_i} \), for some \( e_{\phi_i} \in L_{\phi_i}, e_{-\phi_i} \in L_{-\phi_i} \), the \( \phi_i \) being a fundamental system of simple roots for \( L^* \) relative to \( H \). If \( \phi \) and \( \psi \) are roots, \( \phi \neq -\psi \), then \( L_\phi L_\psi = L_{\phi+\psi} \) if \( \phi + \psi \) is a root, and 0 otherwise. \( l \) and \( l' \) will be said to be conjugates of one another if \( l' \) is the image of \( l \) under an automorphism of \( L^* \). The difference \( r - q \), where \( r, q \) are roots of \( L^* \), but \( r - (r+1) \chi \) and \( q + (q+1) \chi \) are not roots, is denoted by \( A_{r,q} \), and \( A_{\phi,\psi} \) is abbreviated \( A_{ij} \). The algebras of types \( A_1, B_n, C_n, F_4, \) and \( G_2 \) are not included in the list of classical Lie algebras if \( K \) is of characteristic \( p = 2 \). \( G_2 \) is also excluded if \( p = 3 \). The excluded algebras either are not simple or are isomorphic to other classical

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algebras. Because of these exclusions, if $x \neq \pm \psi$, $A_x \equiv 0 \pmod{p}$ if and only if $A_{x^\psi} = 0$.

The aforementioned proof by Shoda was based on two lemmas, namely that any matrix all of whose diagonal elements are zero is expressible as a commutator, and secondly that any matrix of trace zero is similar to a matrix all of whose diagonal elements are zero. Such a matrix is an element of a Lie algebra of type $A_n$ belonging to the subspace $E$. We shall use this fact to rephrase Shoda’s lemmas in the terminology of Lie algebra theory as follows:

**Lemma I.** There exists an $h \in H$ such that $\{ hl \mid l \in L \} = E$.

**Lemma II.** Every element of $L$ has a conjugate in $E$.

For any algebra $L$ for which Lemma I and Lemma II are valid, we can prove

**Theorem A.** Every element $s$ of $L$ can be expressed as a Lie product.

**Proof.** There exists an automorphism $g$ of $L$ such that $g(s) = hl$. Hence $s = g^{-1}(hl)$.

We shall prove that Lemma I is valid for all classical algebras provided that the cardinality of $K$ exceeds $c$ where $c = 2n - 1$ for $A_n$, $4n - 5$ for $B_n$ and $C_n$, $4n - 7$ for $D_n$, $5$ for $G_2$, $15$ for $F_4$, $21$ for $E_6$, $33$ for $E_7$, and $57$ for $E_8$.

Lemma II will be proved valid for all classical Lie algebras.

Theorem A is therefore valid for all classical algebras for which Lemma I is valid.

Let $L = L_1 \oplus \cdots \oplus L_n$. Suppose that Theorem A is valid for $L_i$ for all $i$. Then $s \in L$ can be written $s = s_1 + \cdots + s_n = x_1y_1 + \cdots + x_ny_n = (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)$, and Theorem A is also valid for $L$.

3. **Proof of Lemma I.** Consider an element $a = \sum_\phi m_\phi e_\phi$. It is necessary to find an $h \in H$ such that $h(n_\phi e_\phi) = m_\phi e_\phi$ can be solved for $n_\phi$ for all $\phi$. Let $h = \sum_{i=1}^n q_i h_i$. Then $h(n_\phi e_\phi) = (\sum_i q_i \phi(h_i)) n_\phi e_\phi$. Therefore $h(n_\phi e_\phi) = m_\phi e_\phi$ can be solved for $n_\phi$ if $\sum_i q_i \phi(h_i) \neq 0$. Let $\sum_i \phi(h_i)x_i = g_\phi(x_1, \cdots, x_n)$, a polynomial in $n$ variables. Since there exists an $h_i$ such that $\phi(h_i) \neq 0$, $g_\phi$ is not identically zero. Since $g_\phi = -g_{-\phi}$, we note that if $h(q_1, \cdots, q_n) \neq 0$ where $h = \prod_\phi g_\phi$, then $g_\phi(q_1, \cdots, q_n) \neq 0$ for all $\phi$. Such $g_\phi$ exist provided only that the cardinality of $K$ exceeds the degree of $h(x_1, \cdots, x_n)$ in $x_i$ for all $i$. This degree can be calculated and found to be $c$, the number defined above. The validity of Lemma I for $L^*$ implies its validity for $L$. Therefore, with the exceptions stated above, Lemma I is valid for all classical Lie algebras.
4. Proof of Lemma II.

**Lemma 4.1.** If \( a = \sum_{i=1}^{n} m_i h_i + \sum_{\phi} m_{\phi} e_{\phi} \in Z \), then \( a \) has a conjugate \( a' = \sum_{i=1}^{n} m'_i h_i + \sum_{\phi} m'_{\phi} e_{\phi} \) such that \( m'_j = 0 \) for some \( j \).

**Proof.** We let \( x_{\phi}(y) \) be the automorphism of \( L^* \) having the same effect as \( \exp(\text{ad} ye_{\phi}) \) on each generator, except that \( x_{\phi}(y)e_{-\phi} = e_{-\phi} + y h_{\phi} + y^2 e_{\phi} \) if \( K \) has characteristic 2.

Suppose that there exists a root \( \psi \) such that \( m_{\psi} \neq 0 \). Let \( a' = \sum_{i=1}^{n} m'_i h_i + \sum_{\phi} m'_{\phi} e_{\phi} = x_{-\psi}(y)a \). Then

\[
A_{\psi j} m'_j = \frac{A_{\psi j}}{A_{\psi \psi}} m_{\psi} y,
\]

where \( \psi = \sum_{i=1}^{n} k_i \phi_i \). It is easily observed that \( k_j(A_{\psi j}/A_{\psi \psi}) \equiv 0 \) (mod \( p \)) does not hold for every \( j \). Therefore there is a \( j \) such that \( m'_j = 0 \) can be solved for \( y \), and consequently Lemma 4.1 is valid if \( a \in H \).

Now suppose \( m_{\phi} = 0 \) for all \( \phi \neq 0 \), i.e., \( a \in H \). If it can be shown that \( a \) is conjugate to an element \( a' \in H \), then the lemma will follow by the transitivity of the conjugacy relation. Since \( x_{\phi}(1)a' = \sum m'_i h_i + \sum m'_{\phi} e_{\phi} \) where \( m'_{\phi} = (\sum_{i=1}^{n} m_i h_i)e_{\phi} = a e_{\phi}, m'_j = 0 \) for all \( \phi \) only if \( a e_{\phi} = 0 \) for all \( \phi \). Since \( a \in H \), and \( H \) is abelian, \( a h = 0 \) for \( h \in H \). Thus the Lie product of \( a \) with any generator would be zero, implying \( a \in Z \), thus contradicting the hypothesis of the lemma.

**Lemma 4.2.** Let \( S \) be an indecomposable subset of the fundamental system of roots for a classical Lie algebra \( L^* \). Then the subalgebra \( L^*_S \) generated by \( \{ e_{\phi}, e_{-\phi} \} \) for \( \phi_i \in S \) is a classical Lie algebra unless \( K \) is of characteristic 2, and \( S \) consists of only one root.

**Proof.** Let \( L \) be a complex simple Lie algebra with Cartan subalgebra \( H \) and fundamental system of roots \( \phi_1, \cdots, \phi_n \). Let \( S \) be an indecomposable subset of this fundamental system. Let \( L_S \) be the subalgebra of \( L \) generated by \( \{ e_{\phi_i}, e_{-\phi_i} \} \) for \( \phi_i \in S \). Let \( R \) be the linear transformation from the complex vector space \( V_S \) spanned by \( \phi_i \in S \) into the space of linear functionals on \( H_S = H \cap L_S \), the space spanned by \( \{ h_i \} \) where \( h_i \in S \), defined by \( R(\lambda)h = \lambda h \) \( H_S \). If \( \lambda = \sum c_i \phi_i \) is in the kernel of \( R \), then \( \sum c_i \phi_i(h_i) = 0 \) for all \( j \) such that \( \phi_j(S) \). Since the matrix \( (\phi_i(h_j)) = (A_{ij}) \) for \( i, j \) such that \( \phi_i, \phi_j \in S \) is a principal submatrix of the Cartan matrix of \( L \), it is nonsingular by [2], and so \( \lambda \) must be zero, and \( R \) is one-to-one. Thus if the root \( \psi \) is in \( V_S \), there is an \( h \in H_S \) such that \( h e_{\psi} = \psi(h)e_{\psi} \neq 0 \), and so \( H_S = \{ h \in L_S : \psi(h) = 0 \} \), i.e. \( H_S \) is a Cartan subalgebra of \( L_S \). Clearly, \( \psi \mid H_S \) is a root of \( H_S \) in \( L_S \), and, conversely, every root of \( H_S \) in \( L_S \)
is the restriction to $H_S$ of a root in $V_S$ since the elements $e_\psi$, $\psi$ a root in $V_S$, form a basis for $E_S = E \cap L_S$, and $L_S = H_S + E_S$. Since $R(\psi - \chi)$ is a root of $H_S$ in $L_S$ if and only if $\psi - \chi \in V_S$ is a root of $H$ in $L$, and since $R$ preserves linear independence, $\{\phi_i \mid H_S: \phi_i \in S\}$ is a fundamental system of simple roots for $L_S$. Therefore since $S$ is indecomposable, $L_S$ is simple.

Let $L_Z$ be the Lie subring of $L$ consisting of all integral linear combinations of the basis $\{e_\phi, h_i\}$. Then $\hat{L} = L_Z \otimes \mathbb{Z} K$ is a Lie algebra over the base field $K$ if $(l_1 \otimes k_1)(l_2 \otimes k_2) = (l_1 l_2 \otimes k_1 k_2)$. A comparison of their multiplication tables, identifying $l \otimes k$ with $kl$, reveals that $\hat{L}$ is isomorphic to the classical algebra $L^*$ defined in §2 except when the type of $L$ and the characteristic of $K$ are specifically excluded by that definition. Similarly, $\hat{L}_S = (L_S) \otimes \mathbb{Z} K$ is classical, with the same exceptions, since $L_S$ is simple. However, it is easily observed that the only algebra among these nonclassical algebras which can be a subalgebra of a classical algebra is the algebra of type $A_1$ over a field of characteristic $2$. This establishes the lemma.

Similarly, if $S$ is decomposable, $L_Z^*$ is a direct sum of classical Lie algebras unless $K$ has characteristic $2$, and $S$ has a maximal indecomposable subset containing only one root.

In order to establish Lemma II, we first prove

**Lemma II'**. Every noncentral element of $L^*$ has a conjugate in $E$.

The proof of Lemma II' will proceed by induction on the number of simple roots. First we observe that if $K$ has characteristic $p > 2$, then Lemma II' is valid for $A_1$ by Lemma 4.1.

If $p = 2$, we establish Lemma II' for $A_2$ as follows: Let $i = 1, j = 2$ or $i = 2, j = 1$. Suppose $a = \sum_i m_i h_i + \sum_\phi m_\phi e_\phi$. Let $m_i \neq 0$. We may assume $m_{\phi_j} \neq 0$ or $m_{-\phi_j} \neq 0$ since if $m_{\phi_j} = m_{-\phi_j} = 0$, $x_{\phi_j}(1)a = a' = \sum m'_i h_i + \sum m'_\phi e_\phi$ where $m'_i \neq 0$ and $m'_\phi = m_\phi$. An appropriate value of $y$ can be found for either $x_{\phi_j}(y)a$ or $x_{-\phi_j}(y)a$ to yield an element with $m_j' = m_i'$. If $m_{\phi_1 + \phi_2} = m_{-\phi_1 - \phi_2} = 0$, $x_{\phi_1 + \phi_2}(1)$ will yield $a'$ with $m_{\phi_1 + \phi_2} \neq 0$ and $m_i' = m_i$. Therefore suppose $a = \sum m_i h_i + \sum m_\phi e_\phi$, $m_i = m_j$, and $m_{\phi_1 + \phi_2} \neq 0$ or $m_{-\phi_1 - \phi_2} \neq 0$. Then for appropriate $y \in K$, $x_{\phi_1 + \phi_2}(y)$ or $x_{-\phi_1 - \phi_2}(y)$ applied to $a$ yields an element such that $m_i' = m_j' = 0$. Hence Lemma II' is valid for $A_2$ over a field $K$ of characteristic $2$.

Assume $a \in L^*$, $a \notin Z$, and that the cardinality of the fundamental system of roots of $L^*$ is $n$. If $K$ has characteristic $p > 2$, assume that Lemma II' has been proved for all algebras $L^*$ with fundamental systems of cardinality $c < n$ where $L^*$ is classical or the direct sum of classical algebras. If $p = 2$, assume that Lemma II' has been proved for all algebras with fundamental systems of cardinality $c$ where
Let $a' = \sum m_i h_i + \sum m_\phi e_\phi$ be a conjugate of $a$ such that the number of $i$ such that $m_i \neq 0$ is minimal. We wish to show $a' \in E$.

Let $S$ be the subset consisting of those roots $\phi_i$ of $L^*$ for which $m_i \neq 0$. It is possible to choose $a'$ in such a way that $\sum m_i h_i$ is not in the center $Z_S$ of $L^*_S$. To show this, write $L^*_S = L^*_S \oplus \cdots \oplus L^*_S$. Suppose $\sum m_i h_i \in Z_S$. Let $\phi_i \in S$, but $A_{\phi_i} \neq 0$ for some $\phi_i \in S_i$, where $S_i$ consists of the roots $\phi_{i_1}, \cdots, \phi_{i_r}$. Since $A_1$ has no center if $p > 2$, and $A_1$ is excluded from the list of classical Lie algebras if $p = 2$, we have $r > 1$. Unless $m_{\phi_1 + \phi_1} = m_{-\phi_1 - \phi_1} = 0$, automorphisms $x_{\phi_1 + \phi_1}(y)$ or $x_{-\phi_1 - \phi_1}(y)$ for appropriate $y$ map $a'$ into an element with $m_i' = 0$, $m'_i \neq 0$, $m'_i = m_i$ for $i \neq k, l$. If $m_{\phi_1 + \phi_1} = m_{-\phi_1 - \phi_1} = 0$, the automorphism $x_{\phi_1 + \phi_1}(1)$ maps $a'$ into an element for which $m_{\phi_1 + \phi_1} \neq 0$, thus satisfying the hypothesis of the preceding statement. ($\sum_i m_i h_i - m_k h_k)e_\psi = -m_k h_k e_\psi = -m_k A_{\phi_k} e_\psi$ for $\psi \in L^*_S$, $m_k \neq 0$ by assumption. $A_{\phi_k} \neq 0$ (mod $p$)). Therefore $\sum m_i' h_i$ summed over $i$ such that $\phi_i \in S - \phi_k$ is not in $Z_S$.

If $S_i$ is a single root $\phi_i$ for all $i$, and $p = 2$, then $L^*_S$ where $A_{\phi_i} \neq 0$ is $A_2$, and it is shown above that any element in $A_2$ has a conjugate in $E \cap A_2$. Since $x_{\phi_1}(y)$ for $\phi \in A_2$ leaves $m'_i = m_i$ where $\phi \in \Phi A_2$, we have obtained a contradiction of the minimality assumption.

For all other situations we proceed as follows. Let $a' \in L^*_S$. Then by induction on the cardinality of the fundamental system, $a'$ is conjugate to an element in $E_S = E \cap L^*_S$. Hence $a' \in E$.

If $a' \in L^*_S$, then $a' \in L^*_S + B$ where $B = \sum L_\phi$, summed over $\phi$ such that $L_\phi \subseteq L^*_S$. If $L_\phi \subseteq L^*_S$ and $L_\psi \subseteq L^*_S$, then $x_{\phi}(y)e_\psi \in B$. Therefore $x_\phi(y)b \in B$ for all $b \in B$.

Now we have $a' = a'' + b$, $a'' \in L^*_S$, $b \in B$. By a sequence of automorphisms $x_\phi(y)$ where $L_\phi \subseteq L^*_S$, $a'$ can be transformed into an element in $E_S + B = E$. This completes the proof of Lemma II'.

Lemma II follows immediately from Lemma II', since any automorphism of $L^*$ induces an automorphism of $L$, and an element in the center of $L^*$ has the image zero in $L$.

References


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