ON COMMUTATORS IN A SIMPLE LIE ALGEBRA[1]

GORDON BROWN

1. **Introduction.** K. Shoda [3] has shown that any $n \times n$ matrix of trace zero over a field of characteristic zero is expressible as an additive commutator. A. Albert and B. Muckenhoupt [1] have extended this result to fields of all characteristics. Rephrasing their result, it is easily seen that every element of a Lie algebra of type A_n can be expressed as a Lie product. It seems natural to ask whether a similar assertion is valid for a wider class of Lie algebras.

The purpose of this paper is to employ a generalization of Shoda's method to show that any element of a classical Lie algebra (defined by R. Steinberg [4]) can be expressed as a Lie product provided only that a certain restrictive assumption on the cardinality of the field K over which the algebra is defined is satisfied. This restriction is made in this paper for the sole purpose of permitting a relatively simple method to be applied to prove the main theorem.

2. **The notation and method of proof.** The classical simple Lie algebras will be denoted by $L = L^*/Z$ where Z is the center of L^*. L^* will also be called classical. K will denote the base field. xy is the Lie product of x and y, H a standard Cartan subalgebra, ϕ runs through the nonzero roots of L^* relative to H $L^* = H + E$, a vector space direct sum, where $E = \sum_\phi L_\phi$. Each L_ϕ is one-dimensional, and H is spanned by $h_i = h_{\phi_i}$ $(i = 1, \cdots, n)$, where $\phi_i(h_i) = 2$, $h_i = e_{\phi_i} e_{-\phi_i}$ for some $e_{\phi_i} \in L_{\phi_i}$, $e_{-\phi_i} \in L_{-\phi_i}$, the ϕ_i being a fundamental system of simple roots for L^* relative to H. If ϕ and ψ are roots, $\phi \neq -\psi$, then $L_\phi L_\psi = L_{\phi+\psi}$ if $\phi + \psi$ is a root, and 0 otherwise. l and l' will be said to be conjugates of one another if l' is the image of l under an automorphism of L^*. The difference $r - q$, where $\psi - r\chi$ and $\psi + q\chi$ are roots of L^*, but $\psi - (r+1)\chi$ and $\psi + (q+1)\chi$ are not roots, is denoted by $A_{\chi\psi}$, and $A_{\phi_i\phi_j}$ is abbreviated A_{ij}. The algebras of types A_1, B_n, C_n, F_4, and G_2 are not included in the list of classical Lie algebras if K is of characteristic $p = 2$. G_2 is also excluded if $p = 3$. The excluded algebras either are not simple or are isomorphic to other classical

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algebras. Because of these exclusions, if $\chi \neq \pm\psi$, $A_{\chi\psi} \equiv 0 \pmod{p}$ if and only if $A_{\chi\psi}=0$.

The aforementioned proof by Shoda was based on two lemmas, namely that any matrix all of whose diagonal elements are zero is expressible as a commutator, and secondly that any matrix of trace zero is similar to a matrix all of whose diagonal elements are zero. Such a matrix is an element of a Lie algebra of type A_n belonging to the subspace E. We shall use this fact to rephrase Shoda's lemmas in the terminology of Lie algebra theory as follows:

LEMMA I. *There exists an $h \in H$ such that $\{hl \mid l \in L\} = E$.*

LEMMA II. *Every element of L has a conjugate in E.*

For any algebra L for which Lemma I and Lemma II are valid, we can prove

THEOREM A. *Every element s of L can be expressed as a Lie product.*

PROOF. There exists an automorphism g of L such that $g(s)=hl$. Hence $s=g^{-1}(h)g^{-1}(l)$.

We shall prove that Lemma I is valid for all classical algebras provided that the cardinality of K exceeds c where c is $2n-1$ for A_n, $4n-5$ for B_n and C_n, $4n-7$ for D_n, 5 for G_2, 15 for F_4, 21 for E_6, 33 for E_7, and 57 for E_8.

Lemma II will be proved valid for all classical Lie algebras.

Theorem A is therefore valid for all classical algebras for which Lemma I is valid.

Let $L=L_1 \oplus \cdots \oplus L_n$. Suppose that Theorem A is valid for L_i for all i. Then $s \in L$ can be written $s=s_1+\cdots+s_n=x_1y_1+\cdots+x_ny_n = (\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i)$, and Theorem A is also valid for L.

3. **Proof of Lemma I.** Consider an element $a=\sum_\phi m_\phi e_\phi$. It is necessary to find an $h \in H$ such that $h(n_\phi e_\phi)=m_\phi e_\phi$ can be solved for n_ϕ for all ϕ. Let $h=\sum_{i=1}^n q_ih_i$. Then $h(n_\phi e_\phi)=(\sum_i q_i\phi(h_i))n_\phi e_\phi$. Therefore $h(n_\phi e_\phi)=m_\phi e_\phi$ can be solved for n_ϕ if $\sum_i q_i\phi(h_i)\neq 0$. Let $\sum_i \phi(h_i)x_i = g_\phi(x_1, \cdots, x_n)$, a polynomial in n variables. Since there exists an h_i such that $\phi(h_i)\neq 0$, g_ϕ is not identically zero. Since $g_\phi=-g_{-\phi}$, we note that if $h(q_1, \cdots, q_n)\neq 0$ where $h=\prod_\phi g_\phi$, then $g_\phi(q_1, \cdots, q_n) \neq 0$ for all ϕ. Such q_i exist provided only that the cardinality of K exceeds the degree of $h(x_1, \cdots, x_n)$ in x_i for all i. This degree can be calculated and found to be c, the number defined above. The validity of Lemma I for L^* implies its validity for L. Therefore, with the exceptions stated above, Lemma I is valid for all classical Lie algebras.

4. Proof of Lemma II.

LEMMA 4.1. *If* $a=\sum_{i=1}^{n} m_i h_i + \sum_\phi m_\phi e_\phi \notin Z$, *then* a *has a conjugate* $a'=\sum m_i' h_i + \sum m_\phi' e_\phi$ *such that* $m_j'=0$ *for some* j.

PROOF. We let $x_\phi(y)$ be the automorphism of L^* having the same effect as $\exp(\text{ad } ye_\phi)$ on each generator, except that $x_\phi(y)e_{-\phi}=e_{-\phi}+yh_\phi+y^2 e_\phi$ if K has characteristic 2.

Suppose that there exists a root ψ such that $m_\psi \neq 0$. Let $a'=\sum_{i=1}^{n} m_i' h_i + \sum_\phi m_\phi' e_\phi = x_{-\psi}(y)a$. Then

$$m_j' = m_j + k_j \frac{A_{\psi\phi_j}}{A_{\phi_j\psi}} m_\psi y,$$

where $\psi=\sum_{i=1}^{n} k_i\phi_i$. It is easily observed that $k_j(A_{\psi\phi_j}/A_{\phi_j\psi})\equiv 0 \pmod p$ does not hold for every j. Therefore there is a j such that $m_j'=0$ can be solved for y, and consequently Lemma 4.1 is valid if $a\notin H$.

Now suppose $m_\phi=0$ for all $\phi\neq 0$, i.e., $a\in H$. If it can be shown that a is conjugate to an element $a'\notin H$, then the lemma will follow by the transitivity of the conjugacy relation. Since $x_\phi(1)a$ is $a'=\sum m_i' h_i + \sum m_\phi' e_\phi$ where $m_\phi' e_\phi = (\sum_{i=1}^{n} m_i h_i)e_\phi = ae_\phi$, $m_\phi'=0$ for all ϕ only if $ae_\phi=0$ for all ϕ. Since $a\in H$, and H is abelian, $ah=0$ for $h\in H$. Thus the Lie product of a with any generator would be zero, implying $a\in Z$, thus contradicting the hypothesis of the lemma.

LEMMA 4.2. *Let* S *be an indecomposable subset of the fundamental system of roots for a classical Lie algebra* L^*. *Then the subalgebra* L_S^* *generated by* $\{e_{\phi_i}, e_{-\phi_i}\}$ *for* $\phi_i\in S$ *is a classical Lie algebra unless* K *is of characteristic* 2, *and* S *consists of only one root.*

PROOF. Let L be a complex simple Lie algebra with Cartan subalgebra H and fundamental system of roots $\phi_1, \cdots, \phi_n$. Let S be an indecomposable subset of this fundamental system. Let L_S be the subalgebra of L generated by $\{e_{\phi_i}, e_{-\phi_i}\}$ for $\phi_i\in S$. Let R be the linear transformation from the complex vector space V_S spanned by $\phi_i\in S$ into the space of linear functionals on $H_S=H\cap L_S$, the space spanned by $\{h_i\}$ where $\phi_i\in S$, defined by $R(\lambda)=\lambda|H_S$. If $\lambda=\sum c_i\phi_i$ is in the kernel of R, then $\sum c_i\phi_i(h_j)=0$ for all j such that $\phi_j\in S$. Since the matrix $(\phi_i(h_j))=(A_{ij})$ for i, j such that $\phi_i, \phi_j\in S$ is a principal submatrix of the Cartan matrix of L, it is nonsingular by [2], and so λ must be zero, and R is one-to-one. Thus if the root ψ is in V_S, there is an $h\in H_S$ such that $he_\psi=\psi(h)e_\psi\neq 0$, and so $H_S=\{l\in L_S: lh=0 \text{ for all } h\in H_S\}$, i.e. H_S is a Cartan subalgebra of L_S. Clearly, $\psi|H_S$ is a root of H_S in L_S, and, conversely, every root of H_S in L_S

is the restriction to H_S of a root in V_S since the elements e_ψ, ψ a root in V_S, form a basis for $E_S = E \cap L_S$, and $L_S = H_S + E_S$. Since $R(\psi-\chi)$ is a root of H_S in L_S if and only if $\psi - \chi \in V_S$ is a root of H in L, and since R preserves linear independence, $\{\phi_i | H_S : \phi_i \in S\}$ is a fundamental system of simple roots for L_S. Therefore since S is indecomposable, L_S is simple.

Let L_Z be the Lie subring of L consisting of all integral linear combinations of the basis $\{e_\phi, h_i\}$. Then $\hat{L} = L_Z \otimes_Z K$ is a Lie algebra over the base field K if $(l_1 \otimes k_1)(l_2 \otimes k_2) = (l_1 l_2 \otimes k_1 k_2)$. A comparison of their multiplication tables, identifying $l \otimes k$ with kl, reveals that $\hat{L}$ is isomorphic to the classical algebra L^* defined in §2 except when the type of L and the characteristic of K are specifically excluded by that definition. Similarly, $\hat{L}_S = (L_S)_Z \otimes_Z K$ is classical, with the same exceptions, since L_S is simple. However, it is easily observed that the only algebra among these nonclassical algebras which can be a subalgebra of a classical algebra is the algebra of type A_1 over a field of characteristic 2. This establishes the lemma.

Similarly, if S is decomposable, L_S^* is a direct sum of classical Lie algebras unless K has characteristic 2, and S has a maximal indecomposable subset containing only one root.

In order to establish Lemma II, we first prove

LEMMA II′. *Every noncentral element of L^* has a conjugate in E.*

The proof of Lemma II′ will proceed by induction on the number of simple roots. First we observe that if K has characteristic $p>2$, then Lemma II′ is valid for A_1 by Lemma 4.1.

If $p=2$, we establish Lemma II′ for A_2 as follows: Let $i=1, j=2$ or $i=2, j=1$. Suppose $a = \sum_1^2 m_k h_k + \sum_\phi m_\phi e_\phi$. Let $m_i \neq 0$. We may assume $m_{\phi_j} \neq 0$ or $m_{-\phi_j} \neq 0$ since if $m_{\phi_j} = m_{-\phi_j} = 0$, $x_{\phi_j}(1)a = a' = \sum m_k' h_k + \sum m_\phi' e_\phi$ where $m_{\phi_j}' \neq 0$ and $m_i' = m_i$. An appropriate value of y can be found for either $x_{\phi_j}(y)a$ or $x_{-\phi_j}(y)a$ to yield an element with $m_j' = m_i'$. If $m_{\phi_1+\phi_2} = m_{-\phi_1-\phi_2} = 0$, $x_{\phi_1+\phi_2}(1)$ will yield a' with $m_{\phi_1+\phi_2}' \neq 0$ and $m_k' = m_k$. Therefore suppose $a = \sum m_k h_k + \sum m_\phi e_\phi$, $m_i = m_j$, and $m_{\phi_1+\phi_2} \neq 0$ or $m_{-\phi_1-\phi_2} \neq 0$. Then for appropriate $y \in K$, $x_{\phi_1+\phi_2}(y)$ or $x_{-\phi_1-\phi_2}(y)$ applied to a yields an element such that $m_1' = m_2' = 0$. Hence Lemma II′ is valid for A_2 over a field K of characteristic 2.

Assume $a \in L^*$, $a \notin Z$, and that the cardinality of the fundamental system of roots of L^* is n. If K has characteristic $p>2$, assume that Lemma II′ has been proved for all algebras L^* with fundamental systems of cardinality $c<n$ where L^* is classical or the direct sum of classical algebras. If $p=2$, assume that Lemma II′ has been proved for all algebras with fundamental systems of cardinality c where

$2 \leqq c < n$. Let $a' = \sum m_i h_i + \sum m_\phi e_\phi$ be a conjugate of a such that the number of i such that $m_i \neq 0$ is minimal. We wish to show $a' \in E$.

Let S be the subset consisting of those roots ϕ_j of L^* for which $m_j \neq 0$. It is possible to choose a' in such a way that $\sum m_i h_i$ is not in the center Z_S of L_S^*. To show this, write $L_S^* = L_{S_1}^* \oplus \cdots \oplus L_{S_k}^*$. Suppose $\sum m_i h_i \in Z_S$. Let $\phi_l \notin S$, but $A_{kl} \neq 0$ for some $\phi_k \in S_i$, where S_i consists of the roots $\phi_{i_1}, \cdots, \phi_{i_r}$. Since A_1 has no center if $p > 2$, and A_1 is excluded from the list of classical Lie algebras if $p = 2$, we have $r > 1$. Unless $m_{\phi_k+\phi_l} = m_{-\phi_k-\phi_l} = 0$, automorphisms $x_{\phi_k+\phi_l}(y)$ or $x_{-\phi_k-\phi_l}(y)$ for appropriate y map a' into an element with $m_k' = 0$, $m_l' \neq 0$, $m_i' = m_i$ for $i \neq k, l$. If $m_{\phi_k+\phi_l} = m_{-\phi_k-\phi_l} = 0$, the automorphism $x_{\phi_k+\phi_l}(1)$ maps a' into an element for which $m_{\phi_k+\phi_l} \neq 0$, thus satisfying the hypothesis of the preceding statement. $(\sum_{i_1}^{i_r} m_i h_i - m_k h_k) e_\psi = -m_k h_k e_\psi = -m_k A_{\phi_k \psi} e_\psi$ for $\psi \in L^*_{S \sim \phi_k}$, $m_k \neq 0$ by assumption. $A_{\phi_k \psi} \not\equiv 0 \pmod p$. Therefore $\sum m_i' h_i$ summed over i such that $\phi_i \in S \sim \phi_k$ is not in $Z_{S \sim \phi_k}$.

If S_i is a single root ϕ_i for all i, and $p = 2$, then $L_{S_i \cup \phi_k}$ where $A_{ik} \neq 0$ is A_2, and it is shown above that any element in A_2 has a conjugate in $E \cap A_2$. Since $x_\phi(y)$ for $\phi \in A_2$ leaves $m_l' = m_l$ where $\phi_l \notin A_2$, we have obtained a contradiction of the minimality assumption.

For all other situations we proceed as follows. Let $a' \in L_S^*$. Then by induction on the cardinality of the fundamental system, a' is conjugate to an element in $E_S = E \cap L_S^*$. Hence $a' \in E$.

If $a' \notin L_S^*$, then $a' \in L_S^* + B$ where $B = \sum L_\phi$, summed over ϕ such that $L_\phi \not\subseteq L_S^*$. If $L_\phi \subseteq L_S^*$ and $L_\psi \not\subseteq L_S^*$, then $x_\phi(y) e_\psi \in B$. Therefore $x_\phi(y) b \in B$ for all $b \in B$.

Now we have $a' = a'' + b$, $a'' \in L_S^*$, $b \in B$. By a sequence of automorphisms $x_\phi(y)$ where $L_\phi \subseteq L_S^*$, a' can be transformed into an element in $E_S + B = E$. This completes the proof of Lemma II'.

Lemma II follows immediately from Lemma II', since any automorphism of L^* induces an automorphism of L, and an element in the center of L^* has the image zero in L.

References

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Cornell University