CONCERNING A THEOREM OF L. K. HUA
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1. Denote by Sp(2n) the group of all 2n by 2n matrices of rational integers which satisfy

\[ XHX^T = H, \quad H = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad X^T = X \text{ transpose}. \]

This is the symplectic modular group and in [1] Hua and Reiner show that Sp(2n) may be generated by two matrices for \( n = 1 \), and by four matrices for \( n > 1 \). In this paper we improve their result and prove

**Theorem.** Sp(2n) is generated by three matrices for \( n = 2 \) and \( n = 3 \), and by two matrices for \( n > 3 \).

2. We define the following types of symplectic matrices:

(i) rotations
\[
\left( \begin{array}{cc} A^T & 0 \\ 0 & A^{-1} \end{array} \right), \quad \det A = \pm 1,
\]

(ii) translations
\[
\left( \begin{array}{cc} I & S \\ 0 & I \end{array} \right), \quad S^T = S,
\]

(iii) semi-involutions
\[
\left( \begin{array}{cc} Q & I - Q \\ Q - I & Q \end{array} \right),
\]

where \( Q \) is a diagonal matrix of zeros and ones. Then [1] Sp(2n) is generated by the set of rotations, translations, and semi-involutions. Let \( E_{ij} \) be the \( n \) by \( n \) matrix, all zero except for a one in the \( ij \)th entry. Let \( R_{ij}(x) \) be the rotation, as above, with \( A = I + xE_{ji} \), for \( i \neq j \); \( T_i(x) \) the translation with \( S = xE_{ii} \); and \( T_{ij}(x) \) the translation with \( S = xE_{ij} + xE_{ji} \). Then the \( T \)'s commute and

\[
(T_i(x))^{\pm k} = T_i(\pm kx), \quad (T_{ij}(x))^{\pm k} = T_{ij}(\pm kx), \quad k \text{ any integer}.
\]

If we let \((U, V)\) be the commutator, \( UVU^{-1}V^{-1} \), then

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Now, for \( n > 3 \), \( \text{Sp}(2n) \) is generated by \( J = R_{21}(1)T_n(1) \) and \( D \):

\[
D = \begin{pmatrix}
\sum_{i=1}^{n-1} E_{i,i+1} & -E_{n1} \\
E_{n1} & \sum_{i=1}^{n-1} E_{i,i+1}
\end{pmatrix}.
\]

We compute

\[
(J, D^{-1}JD) = R_{21}(-1).
\]

For indices \( i \) and \( j \), \( n \geq i > j \geq 1 \),

\[
D^{-k}R_{ij}(x)D^k = R_{i-k,j+k}(x), \quad 0 \leq k \leq n - i;
\]

\[
D^{-1}R_{n,i-n-i}(x)D = (T_{j+n-i+i+1,1}(x))^{T};
\]

\[
D^{-k}(T_{j+n-i+i+1,1}(x))^{T}D^k = (T_{j+n-i+i+1+k,1+k}(x))^{T}, \quad 0 \leq k \leq i - 1 - j;
\]

\[
D^{-1}(T_{n,i-i}(x))^{T}D = R_{1,i-f+1}(-x);
\]

\[
D^{-k}R_{1,i-i+1}(-x)D^k = R_{1+k,i-i+1+k}(-x), \quad 0 \leq k \leq n + j - i - 1;
\]

\[
D^{-1}R_{n+i-j,n}(x)D = T_{n+j-i+1,1}(-x);
\]

\[
D^{-k}T_{n+j-i+1,1}(-x)D^k = T_{n+j-i+i+1+k,1+k}(-x), \quad 0 \leq k \leq i - j - 1;
\]

\[
D^{-1}T_{n,i-j}(-x)D = R_{i-j+1,1}(x);
\]

and

\[
D^{1-i}R_{i-j+1,1}(x)D^{i-1} = R_{ij}(x).
\]

Hence \( R_{12}(1) \) is obtained from \( D \) and \( J \).

\[
(J, R_{13}(1)) = R_{23}(1).
\]

Equations (6)–(14) show \( D \) and \( J \) generate every \( R_{i,i+1}(1), R_{i+1,i}(1) \).

Repeated use of (3) and (4) will show every \( R_{ij}(k), i \neq j, k \) any integer is obtained. Hence, the group generated by \( D \) and \( J \) contains every rotation, as above, with \( \det A = 1 \).

\[
JR_{21}(-1) = T_n(1);
\]

and

\[
D^{n-1}T_n(1)D^{1-n} = T_1(1).
\]

\[
D^{-1}T_n(1)D = (T_1(-1))^{T} = P.
\]
(19) \[ T_1(1)PT_1(1) = \begin{pmatrix} I - E_{11} & E_{11} \\ -E_{11} & I - E_{11} \end{pmatrix} = S_1. \]

But \( S_1^2 \) is a rotation with \( A = I - 2E_{11} \). Therefore, from \( D \) and \( J \) any rotation may be had. If we let \( S_{i,j,k,\ldots} \) be the semi-involution where \( Q \) has zeros in the \( i \)th, \( j \)th, \( k \)th, \( \ldots \), positions and ones in the other diagonal positions, then

(20) \[ (S_{i,j,k,\ldots})(S_{i_1,j_1,k_1,\ldots}) = S_{i,i_1,j,j_1,k,k_1,\ldots}. \]

Since

(21) \[ D^{-k}S_1D^k = S_{1+k}, \quad 0 \leq k \leq n - 1, \]

clearly all semi-involutions are available.

To obtain all translations,

(22) \[ D^{-1}R_{1a}(k)D = T_{12}(k). \]

Since

(23) \[
\begin{pmatrix}
I & S \\
0 & I
\end{pmatrix}
\begin{pmatrix}
I & S^T \\
0 & I
\end{pmatrix}
= \begin{pmatrix}
I & S + S^T \\
0 & I
\end{pmatrix},
\]

\[
\begin{pmatrix}
U & 0 \\
0 & (UT)^{-1}
\end{pmatrix}
\begin{pmatrix}
I & S \\
0 & I
\end{pmatrix}
\begin{pmatrix}
U^{-1} & 0 \\
0 & U^T
\end{pmatrix}
= \begin{pmatrix}
I & USU^T \\
0 & I
\end{pmatrix},
\]

by simultaneously interchanging rows and corresponding columns of the symmetric matrices \( kE_{11} \) and \( kE_{22} + kE_{21} \) of \( T_1(k) \) and \( T_{12}(k) \), respectively, every translation is available. This completes the proof for \( n > 3 \).

3. For \( n = 2 \) or \( 3 \), it is now easy to see that the matrices \( D, R_{1a}(1) \), and \( T_1(1) \) will generate \( \text{Sp}(2n) \).

Reference