AN INTEGRAL INEQUALITY

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1. **Introduction.** The purpose of this note is to derive some integral inequalities. In particular, we give conditions on real-valued integrable functions h, g and ϕ defined for all $x \in A$ which imply that

$$\int_A g\phi dx \geqq \int_A h\phi dx \tag{1}$$

or equivalently

$$\int_A (g-h)\phi dx = \int_A f\phi dx \geqq 0$$

where we set $f = g - h$. We show that these results are a generalization of an inequality due to P. R. Beesack [1] except for certain integrability restrictions which he does not require. We use our results to obtain a comparison theorem for the lowest eigenvalue of a membrane. The method used in deriving the inequality (1) also yields a generalization of certain mean value theorems for integrals.

All of our results are obtained by use of the following

LEMMA. *Let f and ϕ be real-valued functions defined for $x \in A$ with f integrable over A. Let ϕ be measurable over A and satisfy the condition $-\infty < m \leqq \phi(x) \leqq M < \infty$. Define the sets*

$$A(y) = \{x : \phi(x) \geqq y\}$$

and

$$B(y) = A - A(y) = \{x : \phi(x) < y\}.$$

Then

$$\int_A f\phi dx = m \int_A f dx + \int_m^M \left(\int_{A(y)} f dx \right) dy \tag{2}$$

and

$$\int_A f\phi dx = M \int_A f dx - \int_m^M \left(\int_{B(y)} f dx \right) dy. \tag{3}$$

PROOF. Define the function F with values

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$$F(y) = \begin{cases} \displaystyle\int_{A(y)} fdx, & y \in [m, M); \\ 0, & y = M. \end{cases}$$

It follows that

$$\int_A f\phi dx = -\int_m^M ydF(y) \tag{4}$$

since for any partition $P_n = \{m = y_0 < y_1 < \cdots < y_n = M\}$ with norm $P_n = \delta < \epsilon / \int_A |f| dx$ and $y_k' \in [y_{k-1}, y_k]$ we have

$$\left| \int_A f\phi dx - \sum_{k=1}^n y_k' [F(y_{k-1}) - F(y_k)] \right| \leqq \delta \int_A |f| \, dx < \epsilon.$$

Integrating the right side of (4) by parts, we get

$$\int_A f\phi dx = -yF(y) \Big|_m^M + \int_m^M F(y)dy = m \int_A fdx + \int_m^M \left(\int_{A(y)} fdx \right) dy,$$

and (2) is proved. (3) follows immediately from

$$(M - m) \int_A fdx = \int_m^M \left(\int_{A(y)} fdx + \int_{B(y)} fdx \right) dy$$

if we replace $\int_m^M (\int_{A(y)} fdx) dy$ by its equivalent from (2).

2. **Inequalities.** In the following, we assume that f, ϕ and $f \cdot \phi$ have finite integrals over the set A. Our lemma then implies the following results.

THEOREM 1. *Let* $-\infty < m \leqq \phi(x)$ *for all* $x \in A$ *and let* $\int_{A(y)} fdx \geqq 0$ *for all* $y \in [m, \infty)$. *Then the condition* $m \int_A fdx \geqq 0$ *implies that*

$$\int_A f\phi dx \geqq 0.$$

PROOF. Let

$$\phi_M(x) = \begin{cases} M, & x \in A(M); \\ \phi(x), & x \in A - A(M). \end{cases}$$

It then follows from our hypothesis and (2) that

$$\int_A f\phi_M dx \geqq 0. \tag{5}$$

By the Lebesgue dominated convergence theorem

$$\lim_{M\to\infty}\int_A f\phi_M dx = \int_A f\phi dx.$$

Hence (5) implies the desired result.

By the same reasoning and (3) we may prove

THEOREM 2. *Let $\phi(x) \leqq M < \infty$ for all $x \in A$ and let $\int_{B(y)} f dx \leqq 0$ for all $y \in (-\infty, M]$. Then the condition $M\int_A f dx \geqq 0$ implies that*

$$\int_A f\phi dx \geqq 0.$$

We may combine the results of Theorems 1 and 2 to get

THEOREM 3. *Let $A_1 = \{x : \phi \geqq 0\}$ and $A_2 = A - A_1$. If $\int_{A(y)} f dx \geqq 0$ for all $y \in [0, \infty)$ and $\int_{B(y)} f dx \leqq 0$ for all $y \in (-\infty, 0)$ then*

$$\int_A f\phi dx \geqq 0.$$

PROOF. By Theorem 1, $\int_{A_1} f\phi dx \geqq 0$. By Theorem 2, $\int_{A_2} f\phi dx \geqq 0$. Adding these inequalities we get the desired result.

3. **Remarks.** Note that the conditions on f are given only in terms of the sets $A(y)$ and $B(y)$. These sets may be known even though the function ϕ is not. This is the case if ϕ is symmetric with respect to a point and has either a positive or negative gradient in A.

As a special case of our results, we have the theorem due to Beesack [1] when $\phi \cdot (F-G)$ is integrable.

THEOREM (BEESACK). *Let F, G and ϕ be integrable over A and let $E_1 = \{x: F(x) \leqq G(x)\}$ and $E_2 = \{x: F(x) > G(x)\}$ and suppose*

$$\int_A G dx \leqq \int_A F dx. \tag{6}$$

Then if either

$$0 \leqq \phi(x_1) \leqq \phi(x_2) \tag{7}$$

or

$$\phi(x_1) \leqq 0 \leqq \phi(x_2) \tag{8}$$

for every pair x_1, x_2 such that $x_1 \in E_1$ and $x_2 \in E_2$

$$\int_A \phi[F - G] dx \geqq 0.$$

We show that the hypothesis of this theorem is a special case of that of Theorems 1 and 3. Let $f=F-G$ and let $\bar{y}=\sup_{x\in E_1}\phi(x)$. Then $\phi(x)\geqq\bar{y}$ for all $x\in E_2$. Hence $A(y)\subset E_2$ and therefore

$$\int_{A(y)} fdx = \int_{A(y)} (F-G)dx \geqq 0$$

for all $y>\bar{y}$. For $y\leqq\bar{y}$, we have

$$\int_{A(y)} (F-G)dx = \int_{E_2} (F-G)dx + \int_{E_1\cap A(y)} (F-G)dx.$$

The first integral on the right is positive while the second is negative. If their sum is negative for some value $y=y_1$, then it is negative for all $y\leqq y_1$. But this contradicts condition (6) since $A(m)=A$. If (7) is satisfied then this implies that the hypothesis of Theorem 1 is also true.

If (8) is true then we have a special case of Theorem 3 since $F-G\leqq 0$ in E_1 implies $\int_{B(y)}(F-G)dx\leqq 0$ for $y<0$ and $F-G\geqq 0$ in E_2 implies $\int_{A(y)}(F-G)dx\geqq 0$ for $y>0$.

4. **A comparison theorem.** The following result is typical of a kind that might be derived from our inequality.

THEOREM 4. *Let $p(x, y)$ and $q(x, y)$ be non-negative real continuous functions defined in a simply connected domain D with a piecewise smooth boundary C such that*

$$\iint_D p(x, y)dxdy = \iint_D q(x, y)dxdy.$$

Consider the eigenvalue problems associated with the nonhomogeneous vibrating membrane over D,

$$\nabla^2 u + \lambda p(x, y)u = 0, \qquad u \equiv 0 \text{ on } C, \tag{9}$$

$$\nabla^2 v + \mu q(x, y)v = 0, \qquad v \equiv 0 \text{ on } C. \tag{10}$$

Let $v_1(x, y)$ denote the eigenfunction corresponding to the lowest eigenvalue μ_1 of (10) *and define*

$$A(z) = \{(x, y): [v_1(x, y)]^2 \geqq z\}.$$

If $\iint_{A(z)}(p-q)dxdy\geqq 0$, for all $z\geqq 0$, then

$$\lambda_1 \leqq \mu_1$$

where λ_1 is the lowest eigenvalue of (9).

PROOF. Since we may choose v_1 so that the condition $0 \leqq v_1 \leqq 1$ is satisfied, Theorem 1 and the above conditions imply

$$\iint_D p v_1^2 dxdy \geqq \iint_D q v_1^2 dxdy.$$

Thus we have

$$\mu_1 = \frac{\iint_D (v_{1x}^2 + v_{1y}^2) dxdy}{\iint_D q v_1^2 dxdy} \geqq \frac{\iint_D (v_{1x}^2 + v_{1y}^2) dxdy}{\iint_D p v_1^2 dxdy} \geqq \lambda_1.$$

In terms of a nonhomogeneous vibrating membrane, our theorem says that if the cumulative mass of a membrane with respect to the sets $A(z)$ is greater than that of the other then the first has a lower fundamental tone. We also note that corresponding results hold for problems of different dimensions and for other boundary conditions.

5. **Mean value theorems.** The mean value theorems stated below are a consequence of our lemma. In the following, we assume that the hypothesis of the lemma is satisfied.

THEOREM 5. *If* $0 \leqq \int_{A(y)} f dx \leqq \int_A f dx$ *for all* $y \in [m, M]$ *then there exists a number* $\gamma \in [m, M]$ *such that*

$$\gamma \int_A f dx = \int_A f \phi dx.$$

PROOF. Since $\int_{A(y)} f dx \leqq \int_A f dx$ implies $\int_{A-A(y)} f dx = \int_{B(y)} f dx \geqq 0$, (2) and (3) give the inequalities

$$m \int_A f dx \leqq \int_A f \phi dx \leqq M \int_A f dx.$$

If $\int_A f dx = 0$, $\int_A f \phi dx = 0$ and the theorem is trivially true; if $\int_A f dx > 0$, then $\gamma = \int_A f \phi dx / \int_A f dx$.

THEOREM 6. *If* $F(y) = \int_{A(y)} f dx$ *is continuous then there is an* $\eta \in [m, M]$ *such that*

$$\int_A f \phi dx = M \int_{A(\eta)} f dx + m \int_{B(\eta)} f dx.$$

PROOF. Applying the one dimensional mean value theorem to the last integral of (3) we get

$$\int_A f\phi dx = M\int_A fdx - \int_{B(\eta)} fdx(M - m)$$
$$= M\int_{A(\eta)} fdx + m\int_{B(\eta)} fdx.$$

We remark that the hypothesis of Theorem 5 replaces the condition $f \geqq 0$ in the classical first mean value theorem for the Lebesgue integral (p. 26 of [2]). Theorem 6 is a generalization of the second mean value theorem for the Lebesgue integral since we do not require a monotonicity condition on ϕ (p. 104 of [2]).

References

1. P. R. Beesack, *A note on an integral inequality*, Proc. Amer. Math. Soc. **8** (1957), 875–879.

2. S. Saks, *Theory of the integral*, Warsaw, 1937.

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