1. Let $\mathbb{N}$ be the set of positive integers, let $E$ be the Banach space of all bounded real-valued functions on $\mathbb{N}$, and let $E^+$ be its positive cone, i.e., $E^+ = \{f \in E | f(n) \geq 0 \text{ for all } n \in \mathbb{N} \}$. If $f - g \in E^+$ we shall say $f \geq g$. If $J \subseteq \mathbb{N}$, we say $J$ has density $a$, or $d(J) = a$, if

$$\lim_n \frac{1}{n} C\{J \cap \{1, 2, \ldots, n\}\} = a.$$  

(Here and henceforth $C(S)$ means the number of elements in $S$.) The intimate connection between density and Cesàro summability is seen in the fact that (1.1) is identical with

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \chi_J(j) = a,$$

where $\chi_J$ is the characteristic function of $J$. The following well-known (e.g. [2, p. 38]) theorem is therefore not surprising.

**Theorem.** Let $f \in E^+$. Then $\lim_n \frac{1}{n} \sum_{j=1}^{n} f(j) = 0$ (i.e., $f$ is $(C, 1)$ summable to 0) if and only if there exists a partition $\mathbb{N} = J \cup K$ such that $d(J) = 0$ and $\lim_{n \in K} f(n) = 0$.

2. The following concept of $F$-summability, or almost-convergence, is due to Lorentz [4].

**Definition 1.** A bounded sequence $f$ is $F$-summable to $a$ if

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} f(j + k) = a,$$

uniformly in $k$.

The above definition reveals that $F$-summability seems to be related to the $(C, 1)$ method. Also, there is clearly a notion of density which corresponds $F$-summability in just the way that ordinary density corresponds to $(C, 1)$ summability; in agreement with the notations of [1; 5], where it is used extensively, we call it $\tau$-density.

**Definition 2.** Let $J \subseteq \mathbb{N}$. We say $d_\tau(J) = a$ if

$$\lim_n \frac{1}{n} C\{J \cap \{k+1, k+2, \ldots, k+n\}\} = a,$$

uniformly in $k$, or, equivalently,

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \chi_J(j + k) = a,$$

uniformly in $k$.

One is now tempted to conjecture a strict analogy to the theorem of §1, with $F$-summability and $\tau$-density replacing $(C, 1)$ summability and ordinary density. In the present paper we prove the falsity of such a conjecture; more precisely, we prove

**Theorem 1.** Let $f \in E^+$. Then the statement

$$\text{(2.4) There exists a partition } \mathbb{N} = J \cup K \text{ such that } d_\tau(J) = 0 \text{ and}$$

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1 This research was supported in part by the National Science Foundation (NSF G-13987).
\( \lim_{n \to \infty} f(n) = 0 \) implies the statement
\[
\lim_{n} \left( \frac{1}{n} \sum_{j=1}^{n} f(j+k) \right) = 0 \text{ uniformly in } k;
\]
but the reverse implication is false.

In fact it will be shown that even (4.5) below, which is a weaker assumption than (2.4), implies (2.5), but not conversely. In §3 we shall give a characterization of \( \tau \)-density in terms of Banach limits, and use it in §§4 and 5 to prove the result in Theorem 1.

3. Let \( E' \) be the conjugate space of \( E \). If \( \phi' \in E' \) and \( f \in E \), \( (\phi', f) \) denotes the value of \( \phi' \) at \( f \). We shall call \( \phi' \) a Banach limit if
\[
(3.1) \| \phi' \| = 1;
\]
\[
(3.2) (\phi', u) = 1, \text{ where } u \in E \text{ is given by } u(n) = 1 \text{ for all } n \in N;
\]
\[
(3.3) (\phi', T_f - f) = 0 \text{ for all } f \in E, \text{ where } T \text{ is the translation operator, } (T_f)(n) = f(n+1) \text{ for all } n \in N.
\]

We denote by \( M' \) the set of all Banach limits; \( M' \) is convex and weak-* compact and nonempty [3]. For any \( f \in E \), the set \( (M', f) \), i.e., \( \{(\phi', f) \mid \phi' \in M' \} \), is a closed interval; by Lorentz’ theorem [4], \( (M', f) \) reduces to a single point if and only if \( f \) is \( F \)-summable to that number. More generally,

**Lemma 1.**
\[
(3.4) \max (M', f) = \limsup_{n} \sup_{k} \left( \frac{1}{n} \sum_{j=1}^{n} f(j+k) \right); \text{ and}
\]
\[
(3.5) \min (M', f) = \liminf_{n} \inf_{k} \left( \frac{1}{n} \sum_{j=1}^{n} f(j+k) \right).
\]

Lemma 1 is a combination of the three lemmas of §2 of [1], for the case of the translation mapping. Equivalent expressions for the bounds of \( (M', f) \) are given in [3] and [4].

**Definition 3.** Let \( J \subseteq N \). Then
\[
(3.6) \Delta_{r}(J) = \max (M', x_{J});
\]
\[
(3.7) \delta_{r}(J) = \min (M', x_{J}).
\]

Using Lemma 1, (3.6) and (3.7) may be rewritten
\[
(3.8) \Delta_{r}(J) = \limsup_{n} \sup_{k} \left( \frac{1}{n} \sum_{j=1}^{n} f(j+k) \right) \subseteq \bigcap \{ k+1, k+2, \ldots, k+n \} \};
\]
and
\[
(3.9) \delta_{r}(J) = \liminf_{n} \inf_{k} \left( \frac{1}{n} \sum_{j=1}^{n} f(j+k) \right) \subseteq \bigcap \{ k+1, k+2, \ldots, k+n \} \}.
\]

Comparing these formulas with (2.2), it is clear that \( d_{r}(J) \) exists if and only if \( \Delta_{r}(J) = \delta_{r}(J) \), and it is their common value.

We now gather some simple facts for later reference:

**Lemma 2.**
\[
(3.10) \text{If } f \in E^{+} \text{ and } \phi' \in M', \text{ then } (\phi', f) \geq 0.
\]
\[
(3.11) d_{r}(J) = 0 \text{ if and only if } (\phi', x_{J}) = 0 \text{ for all } \phi' \in M'.
\]
\[
(3.12) \text{If } N_{i} \subseteq N (i = 1, 2, \ldots, r), \text{ and if } d_{r}(N_{i}) = 0 \text{ for each } i, \text{ then } d_{r}(\bigcup_{i=1}^{r} N_{i}) = 0.
\]
(3.13) If \( f \in E \), \( g \in E \), and if \( f(n) = g(n) \) except on a set of \( \tau \)-density zero, then \( (\phi', f) = (\phi', g) \) for all \( \phi' \in M' \).

(3.14) If \( N = J \cup K \) is a partition, then \( \Delta_r(J) + \delta_r(K) = 1 \).

Proof. (3.10) is an easy and often remarked consequence of (3.1) and (3.2). We shall also use it in the form, if \( f \leq g \) and \( \phi' \in M' \), then \( (\phi', f) \leq (\phi', g) \). (3.11) follows from Definition 3 and the demand that upper and lower \( \tau \)-densities be equal.

To prove (3.12), notice first that \( \chi_{UN} \leq \sum \chi_{N'} \). By (3.11), we must show that if \( \phi' \in M' \), then \( (\phi', \chi_{UN}) = 0 \). But by (3.10), \( (\phi', \chi_{UN}) \leq (\phi', \sum \chi_{N'}) = \sum (\phi', \chi_{N'}) \), which is 0 by (3.11) and the hypothesis. Thus \( (\phi', \chi_{UN}) \leq 0 \), but it cannot be less than 0, by (3.10), because \( \chi_{UN} \geq 0 \).

We shall prove (3.13) in the form: if \( h(n) = 0 \) except on the set \( K \subset N \), with \( d_r(K) = 0 \), then \( (\phi', h) = 0 \) for all \( \phi' \in M' \). But \( -\|h\| \chi_K \leq h \leq \|h\| \chi_K \), so that by (3.10) \( -\|h\| \phi', \chi_K) \leq (\phi', h) \leq \|h\| (\phi', \chi_K) \). But from (3.11) we know that \( (\phi', \chi_K) = 0 \).

To prove (3.14): For every \( \phi' \in M' \), \( (\phi', \chi_J) + (\phi', \chi_K) = 1 \) because \( \chi_J + \chi_K = u \), the unit function of (3.2). Since \( \max (M', \chi_J) \) is actually achieved, it is taken on at some \( \phi' \) at which \( \min (M', \chi_K) \) is achieved. (3.14) thus follows from Definition 3.

4. THEOREM 3. If \( f \in E \), (4.1) is equivalent to (4.2), and (4.3) implies them both. If \( f \in E^+ \), all three statements are equivalent:

(4.1) \[ \lim_{n \to \infty} (1/n) \sum_{j=1}^{n} f(j+k) = 0, \text{ uniformly in } k. \]

(4.2) \[ (M', f) = 0. \]

(4.3) For every \( \delta > 0 \), there exists a partition \( N = J \cup K \), with \( d_r(J) = 0 \), such that if \( n \in K \), \( |f(n)| < \delta \).

Proof. The equivalence of (4.1) and (4.2) was proved already by Lorentz; it is the direct result of the comparison of (3.4) and (3.5). To show (4.3) implies (4.2), let \( \delta > 0 \) be given, and \( J \) and \( K \) taken as in (4.3). Put \( g = f \) on \( K \), \( g = 0 \) on \( J \). Then by (3.13), \( (\phi', g) = (\phi', f) \) for all \( \phi' \in M' \). But \( \|g\| \leq \delta \), hence \( |(\phi', g)| \leq \delta \), hence \( |(\phi', f)| \leq \delta \). As this holds for all \( \delta > 0 \), we obtain (4.2).

Finally, if \( f \in E^+ \), we wish to prove (4.2) implies (4.3). Suppose, denying (4.3), there exists \( \delta > 0 \) such that \( J = \{n \in N \mid f(n) > \delta \} \) does not have \( \tau \)-density 0. Then by (3.11), there exists \( \phi' \in M' \) such that \( (\phi', \chi_J) = \alpha > 0 \). But \( f \geq \delta \chi_J \), hence \( (\phi', f) \geq \delta \alpha > 0 \), i.e., (4.2) is denied.

THEOREM 4. Let \( f \in E^+ \). Then (4.4) \( \Rightarrow \) (4.5) \( \Rightarrow \) (4.6).

(4.4) There exists a partition \( N = J \cup K \) such that \( d_r(J) = 0 \) and \( \lim_{n \to K} f(n) = 0 \).
For every $\alpha > 0$, there exists a partition $N = J \cup K$ such that $\Delta_r(J) < \alpha$, and $\lim_{n \in K} f(n) = 0$.

The function $f$ is $F$-summable to 0 (i.e., any of the statements (4.1), (4.2), or (4.3)).

Proof. (4.4) obviously implies (4.5). Given (4.5), and $\phi' \in M'$, and any $\alpha > 0$, it suffices to show $(\phi', f) < \alpha$, for then (4.2) follows. Choose, by (4.5), the partition $N = J \cup K$ such that $\Delta_r(J) < (\alpha/2\|f\|)$, and $\lim_{n \in K} f(n) = 0$. Then put $K = K_1 \cup K_2$, where $K_1$ is a finite set and $f(n) < \alpha/2$ if $n \in K_2$. Since $K_1$ has zero $\tau$-density, $\Delta_r(J \cup K_1) < (\alpha/2\|f\|)$. Now $(\phi', f) \leq \|f\|(\phi', \chi_{J \cup K_1}) + (\alpha/2)(\phi', \chi_{K_2}) \leq \|f\|\Delta_r(J \cup K_1) + \alpha/2 = \alpha$.

5. Theorem 4 shows that (4.5) and certainly therefore (4.4) implies (2.5), and thus the first part of Theorem 1 is proved. We now construct a function $f \in E$ countering the converse assertion. This is achieved as follows.

We first devise the partition $N = \bigcup_i N_i$, where the elements of $N_i$ are most conveniently given by the columns of the following table:

<table>
<thead>
<tr>
<th>$N_1$</th>
<th>$N_2$</th>
<th>$N_3$</th>
<th>$N_4$</th>
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<td>15</td>
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</table>

and so on. Each $N_i$ is a subsequence of $N$ with constant second differences; from (3.8) it is easily calculable that $d_r(N_i) = 0$ for all $i \in N$.

Lemma 3. Let $K \subset N$. If $K \cap N_i$ is finite for each $i \in N$, then $d_r(K) = 0$.

Proof. From (3.9), it suffices to show that for every $n \in N$ there exists $k \in N$ such that $K \cap \{k+1, k+2, \ldots, k+n\}$ is empty. Now, given $n$, the set $\bar{K} = (N_2 \cap K) \cup (N_3 \cap K) \cup \cdots \cup (N_{n+1} \cap K)$ is finite. Let $k$ be chosen (a) greater than all members of $K$, (b) a member of
$N_i$, and (c) large enough so that $k+1 \in N_2$, $k+2 \in N_3$, \ldots, $k+n \in N_{n+1}$. Then $K \cap \{k+1, k+2, \ldots, k+n\}$ is empty, because it is a subset of $\bar{K}$, and yet is composed entirely of integers larger than any member of $\bar{K}$.

**Definition of $f$.** Put $f(n) = 1/i$ if $n \in N_i$.

**Theorem 5.** The function $f$ defined above is in $E^+$ and it is $F$-summable to 0, but it does not satisfy (4.5).

**Proof.** Since each $N_i$ has $\tau$-density zero, the set $\bigcup_i K_i$ has, by (3.12), $\tau$-density zero, for any $k$. Then, if $k > (1/\delta)$, the partition $N = (\bigcup_i K_i) \cup (\bigcup_{i+1} K_i)$ satisfies (4.3). As $f \in E^+$, (4.2) follows, i.e., $f$ is $F$-summable to 0.

To show $f$ does not obey (4.5), it suffices to take $\alpha = 1$ there. Thus suppose $\Delta_\tau(J) < 1$; we shall show that $\lim_{n \in K} f(n) = 0$ is false, where $N = J \cup K$ is a partition. Now, by (3.14), $\delta_\tau(K) > 0$. By Lemma 3, $K \cap N_i$ is infinite for some $i$. But $f(n) = 1/i$ for all $n \in K \cap N_i$; hence $f(n)$ does not converge to 0 on $K$.

**References**


*Cambridge University and University of Rochester*