CONVERGENCE, DENSITY, AND τ-DENSITY OF BOUNDED SEQUENCES[1]

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1. Let N be the set of positive integers, let E be the Banach space of all bounded real-valued functions on N, and let E^+ be its positive cone, i.e., $E^+ = \{f \in E \mid f(n) \geqq 0 \text{ for all } n \in N\}$. If $f - g \in E^+$ we shall say $f \geqq g$. If $J \subset N$, we say *J has density a*, or $d(J) = a$, if

(1.1) $\lim_n (1/n) C\{J \cap \{1, 2, \cdots, n\}\} = a$.

(Here and henceforth $C(S)$ means the number of elements in S.) The intimate connection between density and Cesàro summability is seen in the fact that (1.1) is identical with

(1.2) $\lim_n (1/n) \sum_{j=1}^n \chi_J(j) = a$,

where χ_J is the characteristic function of J. The following well-known (e.g. [2, p. 38]) theorem is therefore not surprising.

THEOREM. *Let $f \in E^+$. Then* $\lim_n (1/n) \sum_{j=1}^n f(j) = 0$ (*i.e., f is* $(C, 1)$ *summable to* 0) *if and only if there exists a partition $N = J \cup K$ such that* $d(J) = 0$ *and* $\lim_{n \in K} f(n) = 0$.

2. The following concept of F-summability, or almost-convergence, is due to Lorentz [4].

DEFINITION 1. A bounded sequence f is F-summable to a if

(2.1) $\lim_n (1/n) \sum_{j=1}^n f(j+k) = a$, uniformly in k.

The above definition reveals that F-summability seems to be related to the $(C, 1)$ method. Also, there is clearly a notion of density which corresponds F-summability in just the way that ordinary density corresponds to $(C, 1)$ summability; in agreement with the notations of [1; 5], where it is used extensively, we call it *τ-density*.

DEFINITION 2. Let $J \subset N$. We say $d_\tau(J) = a$ if

(2.2) $\lim_n (1/n) C\{J \cap \{k+1, k+2, \cdots, k+n\}\} = a$, uniformly in k, or, equivalently,

(2.3) $\lim_n (1/n) \sum_{j=1}^n \chi_J(j+k) = a$, uniformly in k.

One is now tempted to conjecture a strict analogy to the theorem of §1, with F-summability and τ-density replacing $(C, 1)$ summability and ordinary density. In the present paper we prove the falsity of such a conjecture; more precisely, we prove

THEOREM 1. *Let $f \in E^+$. Then the statement*

(2.4) *There exists a partition $N = J \cup K$ such that $d_\tau(J) = 0$ and*

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$\lim_{n\in K} f(n)=0$ *implies the statement*

(2.5) $\lim_n (1/n)\sum_{j=1}^n f(j+k)=0$ *uniformly in* k;
but the reverse implication is false.

In fact it will be shown that even (4.5) below, which is a weaker assumption than (2.4), implies (2.5), but not conversely. In §3 we shall give a characterization of τ-density in terms of Banach limits, and use it in §§4 and 5 to prove the result in Theorem 1.

3. Let E' be the conjugate space of E. If $\phi'\in E'$ and $f\in E$, (ϕ', f) denotes the value of ϕ' at f. We shall call ϕ' a *Banach limit* if

(3.1) $\|\phi'\|=1$;

(3.2) $(\phi', u)=1$, where $u\in E$ is given by $u(n)=1$ for all $n\in N$;

(3.3) $(\phi', Tf-f)=0$ for all $f\in E$, where T is the *translation operator*, $(Tf)(n)=f(n+1)$ for all $n\in N$.

We denote by M' the set of all Banach limits; M' is convex and weak-* compact and nonempty [**3**]. For any $f\in E$, the set (M', f), i.e., $\{(\phi', f)\mid \phi'\in M'\}$, is a closed interval; by Lorentz' theorem [**4**], (M', f) reduces to a single point if and only if f is F-summable to that number. More generally,

LEMMA 1.

(3.4) $\text{Max } (M', f)=\limsup_n \sup_k (1/n)\sum_{j=1}^n f(j+k)$; and

(3.5) $\text{Min } (M', f)=\liminf_n \inf_k (1/n)\sum_{j=1}^n f(j+k)$.

Lemma 1 is a combination of the three lemmas of §2 of [**1**], for the case of the translation mapping. Equivalent expressions for the bounds of (M', f) are given in [**3**] and [**4**].

DEFINITION 3. Let $J\subset N$. Then

(3.6) $\Delta_\tau(J)=\max (M', \chi_J)$;

(3.7) $\delta_\tau(J)=\min (M', \chi_J)$.

Using Lemma 1, (3.6) and (3.7) may be rewritten

(3.8) $\Delta_\tau(J)=\limsup_n \sup_k (1/n)C\{J\cap\{k+1, k+2, \cdots, k+n\}\}$;
and

(3.9) $\delta_\tau(J)=\liminf_n \inf_k (1/n)C\{J\cap\{k+1, k+2, \cdots, k+n\}\}$.
Comparing these formulas with (2.2), it is clear that $d_\tau(J)$ exists if and only if $\Delta_\tau(J)=\delta_\tau(J)$, and it is their common value.

We now gather some simple facts for later reference:

LEMMA 2.

(3.10) *If* $f\in E^+$ *and* $\phi'\in M'$, *then* $(\phi', f)\geqq 0$.

(3.11) $d_\tau(J)=0$ *if and only if* $(\phi', \chi_J)=0$ *for all* $\phi'\in M'$.

(3.12) *If* $N_i\subset N$ $(i=1, 2, \cdots, r)$, *and if* $d_\tau(N_i)=0$ *for each* i, *then* $d_\tau(\bigcup_{i=1}^r N_i)=0$.

(3.13) *If* $f\in E$, $g\in E$, *and if* $f(n)=g(n)$ *except on a set of* τ*-density zero, then* $(\phi', f)=(\phi', g)$ *for all* $\phi'\in M'$.

(3.14) *If* $N=J\cup K$ *is a partition, then* $\Delta_\tau(J)+\delta_\tau(K)=1$.

Proof. (3.10) is an easy and often remarked consequence of (3.1) and (3.2). We shall also use it in the form, if $f\leqq g$, and $\phi'\in M'$, then $(\phi', f)\leqq(\phi', g)$. (3.11) follows from Definition 3 and the demand that upper and lower τ-densities be equal.

To prove (3.12), notice first that $\chi_{\cup N_i}\leqq\sum\chi_{N_i}$. By (3.11), we must show that if $\phi'\in M'$, then $(\phi', \chi_{\cup N_i})=0$. But by (3.10), $(\phi', \chi_{\cup N_i})\leqq(\phi', \sum\chi_{N_i})=\sum(\phi', \chi_{N_i})$, which is 0 by (3.11) and the hypothesis. Thus $(\phi', \chi_{\cup N_i})\leqq 0$, but it cannot be less than 0, by (3.10), because $\chi_{\cup N_i}\geqq 0$.

We shall prove (3.13) in the form: if $h(n)=0$ except on the set $K\subset N$, with $d_\tau(K)=0$, then $(\phi', h)=0$ for all $\phi'\in M'$. But $-\|h\|\chi_K\leqq h\leqq\|h\|\chi_K$, so that by (3.10) $-\|h\|(\phi', \chi_K)\leqq(\phi', h)\leqq\|h\|(\phi', \chi_K)$. But from (3.11) we know that $(\phi', \chi_K)=0$.

To prove (3.14): For every $\phi'\in M'$, $(\phi', \chi_J)+(\phi', \chi_K)=1$ because $\chi_J+\chi_K=u$, the unit function of (3.2). Since max (M', χ_J) is actually achieved, it is taken on at some ϕ' at which min (M', χ_K) is achieved. (3.14) thus follows from Definition 3.

4. Theorem 3. *If* $f\in E$, (4.1) *is equivalent to* (4.2), *and* (4.3) *implies them both. If* $f\in E^+$, *all three statements are equivalent*:

(4.1) $\lim_n (1/n)\sum_{j=1}^n f(j+k)=0$, *uniformly in* k.

(4.2) $(M', f)=0$.

(4.3) *For every* $\delta>0$, *there exists a partition* $N=J\cup K$, *with* $d_\tau(J)=0$, *such that if* $n\in K$, $|f(n)|<\delta$.

Proof. The equivalence of (4.1) and (4.2) was proved already by Lorentz; it is the direct result of the comparison of (3.4) and (3.5). To show (4.3) implies (4.2), let $\delta>0$ be given, and J and K taken as in (4.3). Put $g=f$ on K, $g=0$ on J. Then by (3.13), $(\phi', g)=(\phi', f)$ for all $\phi'\in M'$. But $\|g\|<\delta$, hence $|(\phi', g)|<\delta$, hence $|(\phi', f)|<\delta$. As this holds for all $\delta>0$, we obtain (4.2).

Finally, if $f\in E^+$, we wish to prove (4.2) implies (4.3). Suppose, denying (4.3), there exists $\delta>0$ such that $J=\{n\in N|f(n)>\delta\}$ does not have τ-density 0. Then by (3.11), there exists $\phi'\in M'$ such that $(\phi', \chi_J)=\alpha>0$. But $f\geqq\delta\chi_J$, hence $(\phi', f)\geqq\delta\alpha>0$, i.e., (4.2) is denied.

Theorem 4. *Let* $f\in E^+$. *Then* (4.4)$\Rightarrow$(4.5)$\Rightarrow$(4.6).

(4.4) *There exists a partition* $N=J\cup K$ *such that* $d_\tau(J)=0$ *and* $\lim_{n\in K} f(n)=0$.

(4.5) *For every* $\alpha>0$, *there exists a partition* $N=J\cup K$ *such that* $\Delta_\tau(J)<\alpha$, *and* $\lim_{n\in K} f(n)=0$.

(4.6) *The function f is F-summable to* 0 (*i.e., any of the statements* (4.1), (4.2), *or* (4.3)).

PROOF. (4.4) obviously implies (4.5). Given (4.5), and $\phi'\in M'$, and any $\alpha>0$, it suffices to show $(\phi',f)<\alpha$, for then (4.2) follows. Choose, by (4.5), the partition $N=J\cup K$ such that $\Delta_\tau(J)<(\alpha/2\|f\|)$, and $\lim_{n\in K} f(n)=0$. Then put $K=K_1\cup K_2$, where K_1 is a finite set and $f(n)<\alpha/2$ if $n\in K_2$. Since K_1 has zero τ-density, $\Delta_\tau(J\cup K_1)<(\alpha/2\|f\|)$. Now $(\phi',f)\leqq\|f\|(\phi',\chi_{J\cup K_1})+(\alpha/2)(\phi',\chi_{K_2})\leqq\|f\|\Delta_\tau(J\cup K_1)+\alpha/2=\alpha$.

5. Theorem 4 shows that (4.5) and certainly therefore (4.4) implies (2.5), and thus the first part of Theorem 1 is proved. We now construct a function $f\in E$ countering the converse assertion. This is achieved as follows.

We first devise the partition $N=\bigcup_1^\infty N_i$, where the elements of N_i are most conveniently given by the columns of the following table:

N_1	N_2	N_3	N_4	N_5
1				
2	3			
4	5	6		
7	8	9	10	
11	12	13	14	15

and so on. Each N_i is a subsequence of N with constant second differences; from (3.8) it is easily calculable that $d_\tau(N_i)=0$ for all $i\in N$.

LEMMA 3. *Let* $K\subset N$. *If* $K\cap N_i$ *is finite for each* $i\in N$, *then* $\delta_\tau(K)=0$.

PROOF. From (3.9), it suffices to show that for every $n\in N$ there exists $k\in N$ such that $K\cap\{k+1, k+2, \cdots, k+n\}$ is empty. Now, given n, the set $\tilde{K}=(N_2\cap K)\cup(N_3\cap K)\cup\cdots\cup(N_{n+1}\cap K)$ is finite. Let k be chosen (a) greater than all members of K, (b) a member of

N_1, and (c) large enough so that $k+1\in N_2$, $k+2\in N_3, \cdots, k+n \in N_{n+1}$. Then $K\cap\{k+1, k+2, \cdots, k+n\}$ is empty, because it is a subset of $\tilde{K}$, and yet is composed entirely of integers larger than any member of $\tilde{K}$.

DEFINITION OF f. Put $f(n)=1/i$ if $n\in N_i$.

THEOREM 5. *The function f defined above is in E^+ and it is F-summable to* 0, *but it does not satisfy* (4.5).

PROOF. Since each N_i has τ-density zero, the set $\bigcup_1^k K_i$ has, by (3.12), τ-density zero, for any k. Then, if $k>(1/\delta)$, the partition $N=(\bigcup_1^k K_i)\cup(\bigcup_{k+1}^\infty K_i)$ satisfies (4.3). As $f\in E^+$, (4.2) follows, i.e., f is F-summable to 0.

To show f does not obey (4.5), it suffices to take $\alpha=1$ there. Thus suppose $\Delta_\tau(J)<1$; we shall show that $\lim_{n\in K} f(n)=0$ is false, where $N=J\cup K$ is a partition. Now, by (3.14), $\delta_\tau(K)>0$. By Lemma 3, $K\cap N_i$ is infinite for some i. But $f(n)=1/i$ for all $n\in K\cap N_i$; hence $f(n)$ does not converge to 0 on K.

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