DIVERGENCE OF APPROXIMATING POLYNOMIALS

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I. Introduction. Let \( f \) be continuous and periodic on the interval \([0, 2\pi)\). For each integer \( n \) let \( F_n \) be a set of \( m_n \) (\( m_n \geq 2n + 1 \)) points, equally spaced modulo \( 2\pi \), from the interval \([0, 2\pi)\). Let \( p_n(f, x) = a_0 + \sum_{k=1}^{\ell_n} (a_k \cos kx + b_k \sin kx) \) be the trigonometric polynomial of order \( n \) which approximates \( f \) best on \( F_n \) in the sense of least squares. Let \( ||f|| = \sup_{0 \leq x \leq 2\pi} |f(x)| \). For each continuous \( f \) let \( P_n \) be the operator defined by \( (P_nf)(x) = p_n(f, x) \). It is easy to verify that \( P_n \) is linear and idempotent \( (P_n^2 = P_n) \). Hence, by a theorem of Nikolaev [2, p. 494]

\[
||P_n|| = \sup_{1 \leq n \leq 1} ||P_nf|| \geq \frac{1}{4\sqrt{\pi}} \log n.
\]

Applying the uniform boundedness principle one infers that there exists a continuous function \( f \) such that \( P_nf \) fails to converge to \( f \) uniformly. This, of course, does not prove that there is a point \( x \), and a continuous function \( f \) such that \( \sup_n |(P_nf)(x)| = \infty \). It is our purpose in this paper to investigate this latter question of pointwise divergence. Call a point \( x \) a point of divergence if the above divergence phenomenon occurs at that point. Our two main results can be stated as follows. For each choice of the sets \( F_n \) the set of points of divergence in \([0, 2\pi)\) is the complement of a set of the first category. If the number of points in \( F_n \), namely \( m_n \), satisfies

\[
\lim_{n \to \infty} \sup_{n \geq 1} \frac{m_n}{n} > \frac{\pi}{2\sqrt{2}},
\]

then every point \( x \) is a point of divergence.

The central tool used to prove both these results is the fact that the approximating operators \( P_n \) for equally spaced points are translation invariant for a point \( x_n \), namely \( x_n = 2\pi/m_n \). Indeed, if we denote by \( T_a \) the operator defined by \( (T_af)(x) = f(x + a) = f_a(x) \), \( T_aP_n = P_nT_{x_n} \). In this setting the two results may be stated in the following way. Let \( Q_n \) be a projection from the space \( C \) of continuous periodic functions on \([0, 2\pi)\) onto the space of trigonometric polynomials

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713
of degree \( \leq n \) satisfying \( T_{x_n}Q_n = Q_nT_{x_n} \) where \( x_n = 2\pi/m_n \) and \( m_n \) is a positive integer. Then if \( \sup_m m_n = \infty \), points of divergence for the sequence of projections \( Q_n \) are of the second category. If \( \lim \sup_m m_n/n > \pi/\sqrt{2} \), then every point \( x \) is a point of divergence.

II. We first state and prove the theorem of Nikolaev for the interval \([0, 2\pi)\). The proof is a slight modification of the one in [2, p. 494].

**Theorem 1.** Let \( Q_n \) be a projection of \( C \) onto the trigonometric polynomials of degree \( \leq n \). Then \( \|Q_n\| \geq (1/4\sqrt{\pi}) \log n \).

**Proof.** It is well known [2, p. 88] that for all \( n, x \)

\[
\left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \leq 2\sqrt{\pi}.
\]

Therefore if

\[
f(x) = \frac{\cos x}{n} + \cdots + \frac{\cos nx}{1} - \frac{\cos(n+2)x}{1} - \cdots - \frac{\cos(2n+1)x}{n},
\]

\[
f(x) = 2\sin(n+1)x \sum_{k=1}^{n} \frac{\sin kx}{k};
\]

and \( |f(x)| \leq 4\sqrt{\pi} \).

Since \( Q_n \) is linear and idempotent,

\[
(Q_nf)(x) = \frac{\cos (x-t)}{n} + \cdots + \cos n(x-t)
\]

\[
+ \sum_{k=n+2}^{2n+1} (f_k(x) \cos kt + g_k(x) \sin kt)
\]

where \( f_k, g_k \) are trigonometric polynomials of degree \( \leq n \). Now

\[
\log n \leq \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{2\pi} \int_{0}^{2\pi} (Q_nf)(t)dt
\]

\[
\leq \sup_{0 \leq t \leq 2\pi} \|Q_n\| \|f_n\| \leq \|Q_n\| 4\sqrt{\pi}.
\]

Consequently \( \|Q_n\| \geq (1/4\sqrt{\pi}) \log n \) as required.

Next we verify the translation invariance property for least squares approximating polynomials on sets \( F_n \) of points equally spaced modulo \( 2\pi \). We write \( F_n = \{ x_k^{(n)} \}, k = 1, \ldots, m_n \), and set \( x_k^{(n)} = y_n + 2\pi k/m_n \), where \( m_n \geq 2n + 1 \). Usually we shall abbreviate \( x_k^{(n)} \) to \( x_k \).

**Lemma 1.** If \( P_n \) is the least squares approximating operator with re-
1963] DIVERGENCE OF APPROXIMATING POLYNOMIALS 715

spect to $F_n$, then $T_{n}P_n = P_n T_{n}$ where $\varepsilon_n = 2\pi/m_n$.

PROOF.

$$(P_n f)(x) = \frac{2}{m_n} \sum_{k=1}^{m_n} f(x_k) D_n(x - x_k) \quad \text{where} \quad D_n(x) = \frac{\sin \left( n + \frac{1}{2} \right) x}{2 \sin \frac{1}{2} x}.$$ 

Therefore

$$(T_{n}P_n f)(x) = \frac{2}{m_n} \sum_{k=1}^{m_n} f(x_k) D_n(x + \varepsilon_n - x_k)$$

$$= \frac{2}{m_n} \sum_{k=1}^{m_n} f(\varepsilon_n + x_k) D_n(x - x_k) = (P_n T_{n} f)(x).$$

Theorem 2. For each $n$ let $Q_n$ be a projection from $\mathbb{C}$ onto the trigonometric polynomials of degree $\leq n$. Suppose for infinitely many positive integers $n_i$, there exist integers $m_j$ for which $T_{2\pi/m_j} Q_{n_j} = Q_{n_j} T_{2\pi/m_j}$. If $\sup_j m_j = \infty$, then the set of points of divergence for the sequence $\{Q_n\}$ is the complement of a set of the first category in $[0, 2\pi)$.

PROOF. By further refining the sequence $n_i$ we may assume $\lim_j m_j = \infty$. Suppose the set $S$ of points of convergence is of the second category. If $x \in S$, then $(Q_n f)(x) \to f(x)$ for each $f \in \mathbb{C}$, and by the uniform boundedness principle

$$\sup_n \sup_{t \in S} |(Q_n f)(x)| < \infty.$$ 

Setting $S_k = \{x: \sup_n \sup_{t \in S} |(Q_n f)(x)| \leq k \}$ we have $S_k$ is a closed subset of $[0, 2\pi)$, and $S = \bigcup_{k=1}^{\infty} S_k$. Therefore $S_k$ contains an open interval $I = (\alpha, \beta)$ for some integer $k_0$. Choose $j_0$ large enough so that $j > j_0$ implies $2\pi/m_j < |\beta - \alpha|$. Then $\bigcup_{k=1}^{\infty} ((2\pi/|m_j|) + I)$ covers the interval $[0, 2\pi)$. Now applying the translation invariance hypothesis we have $\sup_{x \in [0, 2\pi]} \sup_{t \in S} |(Q_n g)(x)| \leq k_0$ if $\|f\| \leq 1$. Hence, $\|Q_n f\| \leq k_0$ for infinitely many integers $n_j$ which contradicts the theorem of Nikolaev.

Examples can be given of a sequence of least squares approximating projections $\{P_n\}$ satisfying the conditions of Theorem 2 for which $(P_n f)(x) \to f(x)$ for all $x$ in a countable dense set. For instance let $F_n$ be a sequence of subsets of $[0, 2\pi)$ each containing $2n+1$ points and let $P_n$ be the interpolating projection for $F_n$. Now if one requires $F_n \subset F_{n+1}$ and that the points of $F_n$ are equally spaced when $n = 1 + 3 + \cdots + 3^k$, $k = 1, 2, \cdots$, then $P_n f(x) \to f(x)$ for each $x \in \bigcup F_n$; and $\{P_n\}$ satisfies the hypothesis of Theorem 2.

A natural conjecture in this regard is that the translation invari-
ance hypothesis can be dropped from Theorem 2. The author has been unable to prove this, however.

III. In this section we give a sufficient condition for every point of the interval to be a point of divergence for the sequence \( \{Q_n\} \). We need first a preliminary result due to Bernstein [1, p. 57].

**Lemma 2.** Let \( F_n \) be a finite set of points in \((0, 2\pi)\) such that for each \( x \in (0, 2\pi) \), \( d(x, F_n) = \inf_{y \in F_n} |x - y| \leq \pi/m_n \). If \( p_n \) is a trigonometric polynomial of order \( n \), and \( m_n/n \geq \lambda > \pi/\sqrt{2} \), then

\[
\frac{1}{\pi^2} \leq \frac{2\lambda^2}{2\lambda^2 - \pi^2} \sup_{x \in F_n} |p_n(x)|.
\]

**Proof.** Let \( M = \|p_n\| \), and \( N = \sup_{x \in F_n} |p_n(x)| \). Then by a familiar theorem of Bernstein \( |p_n'(x)| \leq n^2 M \). If \( x_0 \) is a point at which \( \|p_n\| \) is attained, then \( p_n'(x_0) = 0 \); and hence

\[
|p_n'(x)| \leq |x - x_0| n^2 M,
\]

and

\[
|p_n(x) - p_n(x_0)| \leq \frac{|x - x_0|^2}{2} n^2 M.
\]

Since \( d(x_0, F_n) \leq \pi/m_n \), this yields

\[
M - N \leq \frac{\pi^2 n^2}{2m^2_n} M,
\]

or

\[
M \leq \frac{1}{\pi^2 n^2} \frac{2\lambda^2}{2\lambda^2 - \pi^2} N.
\]

**Theorem 3.** For each \( n \) let \( Q_n \) be a projection from \( C \) onto the trigonometric polynomials of degree \( \leq n \). Suppose for infinitely many positive integers \( n \), there exist integers \( m_n \) for which

\[
T_{2\pi/m_n} Q_{n} = Q_{n} T_{2\pi/m_n}.
\]

If \( \lim sup_j m_j/n_j > \pi/\sqrt{2} \), then all points of \([0, 2\pi)\) are points of divergence.

**Proof.** Suppose \( x_0 \) is a point of convergence then

\[\text{Here } |x| \text{ is computed modulo } 2\pi.\]
\[ \sup_{n} \sup_{l/l \leq 1} |(Q_{n}f)(x_{0})| < \infty. \]

But if \( T_{2\pi/m_{n}}Q_{n} = Q_{n}T_{2\pi/m_{n}} \), then

\[ L_{n} = \sup_{l/l \leq 1} |(Q_{n}f)(x_{0})| = \sup_{l/l \leq 1} \left| (Q_{n}f) \left( x_{0} + \frac{2\pi l}{m_{n}} \right) \right|, \quad l = 1, 2, \ldots, m_{n}. \]

Applying Lemma 2 to \( F_{n} = \{ x_{0} + (2\pi l/m_{n}) \} \), \( l = 1, \ldots, m_{n} \), when \( m_{n}/n \geq \lambda > \pi/\sqrt{2} \), we have that \( \| Q_{n} \| \leq \text{const.} \; L_{n} \). The hypothesis of the theorem then implies \( \sup_{l} \| Q_{n} \| < \infty \) for some sequence of integers \( \{ n_{j} \} \). This again contradicts the theorem of Nikolaev.

Some rate of growth assumption for \( m_{n} \) is necessary in order to guarantee that all points of \([0, 2\pi)\) are points of divergence. For, referring back to the example of §11, suppose

\[ n_{k} = 1 + \cdots + 3^{k} \leq n < 1 + \cdots + 3^{k+1} = n_{k+1}. \]

Define

\[ Q_{n} = \frac{1}{2n_{k}+1} \sum_{j=1}^{2n_{k}+1} T_{2\pi j/(2n_{k}+1)}P_{n}T_{2\pi j/(2n_{k}+1)}. \]

It may be verified easily that \( Q_{n} \) is a projection of \( C \) onto the trigonometric polynomials of order \( \leq n \). \( Q_{n} = P_{n} \) if \( n = n_{k} \); and if \( n_{k} \leq n < n_{k+1} \), \( T_{2\pi/(2n_{k}+1)}Q_{n} = Q_{n}T_{2\pi/(2n_{k}+1)} \). Moreover, for \( n \geq n_{k} \), \( (Q_{n}f)(x) = f(x) \) if \( x \in F_{n} \). Therefore \( (Q_{n}f)(x) \rightarrow f(x) \) if \( x \in U_{n}F_{n} \).

**Bibliography**


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