DIVERGENCE OF APPROXIMATING POLYNOMIALS[1]

PHILIP C. CURTIS, JR.

I. **Introduction.** Let f be continuous and periodic on the interval $[0, 2\pi)$. For each integer n let F_n be a set of m_n ($m_n \geqq 2n+1$) points, equally spaced modulo 2π, from the interval $[0, 2\pi)$. Let $p_n(f, x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$ be the trigonometric polynomial of order n which approximates f best on F_n in the sense of least squares. Let $\|f\| = \sup_{0 \leq x \leq 2\pi} |f(x)|$. For each continuous f let P_n be the operator defined by $(P_n f)(x) = p_n(f, x)$. It is easy to verify that P_n is linear and idempotent ($P_n^2 = P_n$). Hence, by a theorem of Nikolaev [2, p. 494]

$$\|P_n\| = \sup_{\|f\| \leqq 1} \|P_n f\| \geqq \frac{1}{4\sqrt{\pi}} \log n.$$

Applying the uniform boundedness principle one infers that there exists a continuous function f such that $P_n f$ fails to converge to f uniformly. This, of course, does not prove that there is a point x, and a continuous function f such that $\sup_n |(P_n f)(x)| = \infty$. It is our purpose in this paper to investigate this latter question of pointwise divergence. Call a point x a point of divergence if the above divergence phenomenon occurs at that point. Our two main results can be stated as follows. For each choice of the sets F_n the set of points of divergence in $[0, 2\pi)$ is the complement of a set of the first category. If the number of points in F_n, namely m_n, satisfies

$$\limsup_n \frac{m_n}{n} > \frac{\pi}{\sqrt{2}},$$

then every point x is a point of divergence.

The central tool used to prove both these results is the fact that the approximating operators P_n for equally spaced points are translation invariant for a point x_n, namely $x_n = 2\pi/m_n$. Indeed, if we denote by T_a the operator defined by $(T_a f)(x) = f(x+a) \equiv f_a(x)$, $T_{x_n} P_n = P_n T_{x_n}$. In this setting the two results may be stated in the following way. Let Q_n be a projection from the space $\mathfrak{C}$ of continuous periodic functions on $[0, 2\pi)$ onto the space of trigonometric polynomials

Presented to the Society, August 31, 1962 under the title *Pointwise divergence of approximating polynomials;* received by the editors June 22, 1962.

[1] This research was supported by the United States Air Force through work done at Space Technology Laboratories, Inc. and under contract AF 49(638)-859.

of degree $\leqq n$ satisfying $T_{x_n}Q_n = Q_n T_{x_n}$ where $x_n = 2\pi/m_n$ and m_n is a positive integer. Then if $\sup_n m_n = \infty$, points of divergence for the sequence of projections Q_n are of the second category. If $\limsup_n m_n/n > \pi/\sqrt{2}$, then every point x is a point of divergence.

II. We first state and prove the theorem of Nikolaev for the interval $[0, 2\pi)$. The proof is a slight modification of the one in [2, p. 494].

THEOREM 1. *Let Q_n be a projection of $\mathfrak{C}$ onto the trigonometric polynomials of degree $\leqq n$. Then* $\|Q_n\| \geqq (1/4\sqrt{\pi}) \log n$.

PROOF. It is well known [2, p. 88] that for all n, x

$$\left| \sum_{k=1}^{n} \frac{\sin kx}{k} \right| \leqq 2\sqrt{\pi}.$$

Therefore if

$$f(x) = \frac{\cos x}{n} + \cdots + \frac{\cos nx}{1} - \frac{\cos(n+2)x}{1} - \cdots - \frac{\cos(2n+1)x}{n},$$

$$f(x) = 2\sin(n+1)x \sum_{k=1}^{n} \frac{\sin kx}{k};$$

and $|f(x)| \leqq 4\sqrt{\pi}$.

Since Q_n is linear and idempotent,

$$\begin{aligned}(Q_n f_{-t})(x) &= \frac{\cos(x-t)}{n} + \cdots + \cos n(x-t) \\ &\quad + \sum_{k=n+2}^{2n+1} (f_k(x)\cos kt + g_k(x)\sin kt)\end{aligned}$$

where f_k, g_k are trigonometric polynomials of degree $\leqq n$. Now

$$\begin{aligned}\log n &\leqq \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{2\pi}\int_0^{2\pi} (Q_n f_{-t})(t)\,dt \\ &\leqq \sup_{0 \leqq t \leqq 2\pi} \|Q_n\| \, \|f_{-t}\| \leqq \|Q_n\| 4\sqrt{\pi}.\end{aligned}$$

Consequently $\|Q_n\| \geqq (1/4\sqrt{\pi}) \log n$ as required.

Next we verify the translation invariance property for least squares approximating polynomials on sets F_n of points equally spaced modulo 2π. We write $F_n = \{x_k^{(n)}\}$, $k = 1, \cdots, m_n$, and set $x_k^{(n)} = y_n + 2\pi k/m_n$, where $m_n \geqq 2n+1$. Usually we shall abbreviate $x_k^{(n)}$ to x_k.

LEMMA 1. *If P_n is the least squares approximating operator with re-*

spect to F_n, then $T_{z_n}P_n = P_n T_{z_n}$ where $z_n = 2\pi/m_n$.

PROOF.

$$(P_n f)(x) = \frac{2}{m_n} \sum_{k=1}^{m_n} f(x_k) D_n(x - x_k) \quad \text{where} \quad D_n(x) = \frac{\sin (n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}.$$

Therefore

$$\begin{aligned}(T_{z_n} P_n f)(x) &= \frac{2}{m_n} \sum_{k=1}^{m_n} f(x_k) D_n(x + z_n - x_k) \\ &= \frac{2}{m_n} \sum_{k=1}^{m_n} f(z_n + x_k) D_n(x - x_k) = (P_n T_{z_n} f)(x).\end{aligned}$$

THEOREM 2. *For each n let Q_n be a projection from $\mathfrak{C}$ onto the trigonometric polynomials of degree $\leqq n$. Suppose for infinitely many positive integers n_j, there exist integers m_j for which $T_{2\pi/m_j}Q_{n_j} = Q_{n_j}T_{2\pi/m_j}$. If $\sup_j m_j = \infty$, then the set of points of divergence for the sequence $\{Q_n\}$ is the complement of a set of the first category in $[0, 2\pi)$.*

PROOF. By further refining the sequence n_j we may assume $\lim_j m_j = \infty$. Suppose the set S of points of convergence is of the second category. If $x \in S$, then $(Q_n f)(x) \to f(x)$ for each $f \in \mathfrak{C}$, and by the uniform boundedness principle

$$\sup_n \sup_{\|f\| \leqq 1} |(Q_n f)(x)| < \infty.$$

Setting $S_k = \{x : \sup_n \sup_{\|f\| \leqq 1} |(Q_n f)(x)| \leqq k\}$ we have S_k is a closed subset of $[0, 2\pi)$, and $S = \bigcup_{k=1}^{\infty} S_k$. Therefore S_{k_0} contains an open interval $I = (\alpha, \beta)$ for some integer k_0. Choose j_0 large enough so that $j > j_0$ implies $2\pi/m_j < |\beta - \alpha|$. Then $\bigcup_{l=1}^{m_j} ((2\pi l/m_j) + I)$ covers the interval $[0, 2\pi)$. Now applying the translation invariance hypothesis we have $\sup_{x \in [0, 2\pi)} \sup_{\|g\| \leqq 1} |(Q_n g)(x)| \leqq k_0$ if $\|f\| \leqq 1$. Hence, $\|Q_{n_j}\| \leqq k_0$ for infinitely many integers n_j which contradicts the theorem of Nikolaev.

Examples can be given of a sequence of least squares approximating projections $\{P_n\}$ satisfying the conditions of Theorem 2 for which $(P_n f)(x) \to f(x)$ for all x in a countable dense set. For instance let $\{F_n\}$ be a sequence of subsets of $[0, 2\pi)$ each containing $2n+1$ points and let P_n be the interpolating projection for F_n. Now if one requires $F_n \subset F_{n+1}$ and that the points of F_n are equally spaced when $n = 1 + 3 + \cdots + 3^k$, $k = 1, 2, \cdots$, then $P_n f(x) \to f(x)$ for each $x \in \bigcup F_n$; and $\{P_n\}$ satisfies the hypothesis of Theorem 2.

A natural conjecture in this regard is that the translation invariance hypothesis can be dropped from Theorem 2. The author has been unable to prove this, however.

III. In this section we give a sufficient condition for every point of the interval to be a point of divergence for the sequence $\{Q_n\}$. We need first a preliminary result due to Bernstein [1, p. 57].

LEMMA 2. *Let F_n be a finite set of points in $[0, 2\pi)$ such that for each $x \in (0, 2\pi)$, $d(x, F_n) \equiv \inf_{y \in F_n} |x-y| \leqq \pi/m_n$.*[2] *If p_n is a trigonometric polynomial of order n, and $m_n/n \geqq \lambda > \pi/\sqrt{2}$, then*

$$\|p_n\| \leqq \frac{2\lambda^2}{2\lambda^2 - \pi^2} \sup_{x \in F_n} |p_n(x)|.$$

PROOF. Let $M = \|p_n\|$, and $N = \sup_{x \in F_n} |p_n(x)|$. Then by a familiar theorem of Bernstein $|p_n''(x)| \leqq n^2 M$. If x_0 is a point at which $\|p_n\|$ is attained, then $p_n'(x_0) = 0$; and hence

$$|p_n'(x)| \leqq |x - x_0| n^2 M,$$

and

$$|p_n(x) - p_n(x_0)| \leqq \frac{|x - x_0|^2}{2} n^2 M.$$

Since $d(x_0, F_n) \leqq \pi/m_n$, this yields

$$M - N \leqq \frac{\pi^2 n^2}{2m_n^2} M,$$

or

$$M \leqq \frac{1}{1 - \dfrac{\pi^2 n^2}{2m_n^2}} N \leqq \frac{2\lambda^2}{2\lambda^2 - \pi^2} N.$$

THEOREM 3. *For each n let Q_n be a projection from $\mathfrak{C}$ onto the trigonometric polynomials of degree $\leqq n$. Suppose for infinitely many positive integers n_j there exist integers m_j for which*

$$T_{2\pi/m_j} Q_{n_j} = Q_{n_j} T_{2\pi/m_j}.$$

If $\limsup_j m_j/n_j > \pi/\sqrt{2}$, *then all points of $[0, 2\pi)$ are points of divergence.*

PROOF. Suppose x_0 is a point of convergence then

[2] Here $|x|$ is computed modulo 2π.

$$\sup_n \sup_{\|f\| \leqq 1} |(Q_n f)(x_0)| < \infty.$$

But if $T_{2\pi/m_n} Q_n = Q_n T_{2\pi/m_n}$, then

$$L_n \equiv \sup_{\|f\| \leqq 1} |(Q_n f)(x_0)| = \sup_{\|f\| \leqq 1} \left| (Q_n f)\left(x_0 + \frac{2\pi l}{m_n}\right)\right|, \quad l = 1, 2, \cdots, m_n.$$

Applying Lemma 2 to $F_n = \{x_0 + (2\pi l/m_n)\}$, $l = 1, \cdots, m_n$, when $m_n/n \geqq \lambda > \pi/\sqrt{2}$, we have that $\|Q_n\| \leqq \text{const. } L_n$. The hypothesis of the theorem then implies $\sup_j \|Q_{n_j}\| < \infty$ for some sequence of integers $\{n_j\}$. This again contradicts the theorem of Nikolaev.

Some rate of growth assumption for m_n is necessary in order to guarantee that all points of $[0, 2\pi)$ are points of divergence. For, referring back to the example of §II, suppose

$$n_k \equiv 1 + \cdots + 3^k \leqq n < 1 + \cdots + 3^{k+1} \equiv n_{k+1}.$$

Define

$$Q_n = \frac{1}{2n_k + 1} \sum_{j=1}^{2n_k+1} T_{2\pi j/(2n_k+1)} P_n T^{-1}_{2\pi j/(2n_k+1)}.$$

It may be verified easily that Q_n is a projection of $\mathfrak{C}$ onto the trigonometric polynomials of order $\leqq n$. $Q_n = P_n$ if $n = n_{k_j}$; and if $n_k \leqq n < n_{k+1}$, $T_{2\pi/(2n_k+1)} Q_n = Q_n T_{2\pi/(2n_k+1)}$. Moreover, for $n \geqq n_k$, $(Q_n f)(x) = f(x)$ if $x \in F_{n_k}$. Therefore $(Q_n f)(x) \to f(x)$ if $x \in \bigcup_n F_n$.

Bibliography

1. S. Bernstein, *Sur une modification de la formule d'interpolation de Lagrange* Comm. Soc. Math. Kharkov **5** (1931), 49–57.

2. I. P. Natanson, *Konstruktive Funktionentheorie*, Akademie Verlag, Berlin, 1955.

University of California, Los Angeles