

ON A RELATED FUNCTION THEOREM

E. B. LEACH

1. In a previous note [1], the strong differential, a variant of the classical Fréchet differential, was defined. Strong differentiability at a point seems to be a good smoothness condition for related function theorems, being stronger than the insufficient condition of Fréchet differentiability at the point and weaker than Fréchet differentiability in a neighborhood of the point, together with continuity of the differential at the point.

In this note we state as a lemma a slight generalization of the theorem of [1]. Algebraic manipulation of the relations involved then enables us to extend the range over which the conclusion of the lemma is valid.

2. The following definition is given in [1].

DEFINITION. Let A and B be open subsets of Banach spaces U and V , respectively, and let $f: A \rightarrow B$ be a function. We say that f has *strong differential* α at a point $x_0 \in A$, if $\alpha: U \rightarrow V$ is a bounded linear transformation and for every $\epsilon > 0$ there is a number $\delta > 0$ such that

$$(1) \quad |f(x'') - f(x') - \alpha(x'' - x')| \leq \epsilon |x'' - x'|,$$

whenever $|x' - x_0| < \delta$ and $|x'' - x_0| < \delta$.

When f has a strong differential α at x_0 we shall write $f'(x_0) = \alpha$. The following lemma is the basic analytical tool of our discussion. (Throughout the rest of this note f will denote a fixed function relating open subsets A and B of Banach spaces U and V .)

LEMMA. Let $\alpha: U \rightarrow V$ and $\beta: V \rightarrow U$ be bounded linear transformations such that $\beta\alpha\beta = \beta$. Let $x_0 \in A$ and $y_0 \in V$ satisfy $\beta(f(x_0)) = \beta(y_0)$ and $\beta(\alpha(x_0)) = x_0$. Finally, suppose that $f'(x_0) = \alpha$. Then there are neighborhoods A_0 of x_0 and B_0 of y_0 (with $A_0 \subset A$) such that:

(i) There is a unique function $g: B_0 \rightarrow A_0$ satisfying $\beta(\alpha(g(y))) = g(y)$ and $\beta(f(g(y))) = \beta(y)$, for all $y \in B_0$.

(ii) g is continuous, $g(y_0) = x_0$, and if for any $y_1 \in B_0$, $f'(g(y_1)) = \alpha_1$, then $g'(y_1) = \gamma^{-1}\beta$, where $\gamma = 1 + \beta(\alpha_1 - \alpha)$.

The neighborhoods A_0 and B_0 are described in terms of an $\epsilon > 0$, chosen so that $\epsilon|\beta| < 1/2$, and a $\delta > 0$, chosen so that δ and ϵ satisfy the condition of differentiability of f at x_0 and so that the sphere of

Presented to the Society, February 23, 1963; received by the editors July 21, 1962.

radius δ about x_0 is in A . Then A_0 is the sphere of radius δ about x_0 and B_0 is the sphere of radius $\delta/2|\beta|$ about y_0 . The function g is the uniform limit of a sequence $\{g_n\}$ of functions defined recursively by:

$$(2) \quad g_0(y) = x_0; \quad g_{n+1}(y) = g_n(y) + \beta(y - f(g_n(y))), \quad \text{if } n \geq 0.$$

The entire proof corresponds exactly with the proof of the theorem of [1], and will be omitted.

Let R be the relation consisting of pairs (x, y) , such that $x \in A$, $y \in V$, $\beta(\alpha(x)) = x$ and $\beta(f(x)) = \beta(y)$. The lemma asserts that near a point (x_0, y_0) of R , points of R consist exactly of the pairs $(g(y), y)$, provided $f'(x_0) = \alpha$. The next theorem extends the range over which this conclusion is valid.

THEOREM. *Let $\alpha: U \rightarrow V$ and $\beta: V \rightarrow U$ be bounded linear transformations such that $\beta\alpha\beta = \beta$. Let $x_1 \in A$ and $y_1 \in V$ be points satisfying $\beta(\alpha(x_1)) = x_1$ and $\beta(f(x_1)) = \beta(y_1)$. Finally, assume $f'(x_1) = \alpha_1$, a transformation such that $\gamma = 1 + \beta(\alpha_1 - \alpha)$ has a bounded inverse. Then there is a pair of neighborhoods A_1 of x_1 and B_1 of y_1 such that:*

(i) *There is a unique function $g: B_1 \rightarrow A_1$ satisfying $\beta(\alpha(g(y))) = g(y)$ and $\beta(f(g(y))) = \beta(y)$, for all $y \in B_1$.*

(ii) *g is continuous, $g(y_1) = x_1$ and if for any $y_2 \in B_1$, $f'(g(y_2)) = \alpha_2$, then $g'(y_2) = [1 + \beta(\alpha_2 - \alpha)]^{-1}\beta$.*

PROOF. For any bounded linear transformation $\alpha_2: U \rightarrow V$, let $\beta_1 = \gamma^{-1}\beta$ and $\gamma_1 = 1 + \beta_1(\alpha_2 - \alpha_1)$. Then γ_1 has a bounded inverse if and only if $\gamma\gamma_1$ has a bounded inverse. In this case, let $\beta_2 = \gamma_1^{-1}\beta_1$. Then $\beta_2 = \gamma_1^{-1}\gamma^{-1}\beta = (\gamma\gamma_1)^{-1}\beta$, where $\gamma\gamma_1 = \gamma[1 + \gamma^{-1}\beta(\alpha_2 - \alpha_1)] = \gamma + \beta(\alpha_2 - \alpha_1) = 1 + \beta(\alpha_1 - \alpha) + \beta(\alpha_2 - \alpha_1) = 1 + \beta(\alpha_2 - \alpha)$. In particular, if $\alpha_2 = \alpha$, $\beta_2 = \beta$, and so the pairs (α, β) and (α_1, β_1) are symmetrically related.

Further, an identity of the type we are considering involving α and β may be replaced by the corresponding identity about α_1 and β_1 . To see this, first note that because $\beta\alpha\beta = \beta$, then $\beta\alpha\gamma = \beta\alpha + \beta\alpha\beta(\alpha_1 - \alpha) = \beta\alpha_1$, and composing with γ^{-1} on the right we obtain:

$$(3) \quad \beta\alpha = \beta\alpha_1\gamma^{-1}.$$

Also, $\gamma\beta = \beta + \beta(\alpha_1 - \alpha)\beta = \beta\alpha_1\beta$, and composing with γ^{-1} on the left we obtain:

$$(4) \quad \beta = \gamma^{-1}\beta\alpha_1\beta.$$

Now $\beta_1\alpha_1\beta_1 = \gamma^{-1}\beta\alpha_1\gamma^{-1}\beta = \gamma^{-1}\beta\alpha\beta = \gamma^{-1}\beta = \beta_1$, using (3) at the second step. For any x , if $\beta(\alpha(x)) = x$, then $\beta_1(\alpha_1(x)) = \gamma^{-1}(\beta(\alpha_1(x))) = \gamma^{-1}(\beta(\alpha_1(\beta(\alpha(x)))) = \beta(\alpha(x)) = x$, using (4) at the third step. Fi-

nally, if $\beta(f(x)) = \beta(y)$, for a pair (x, y) , then $\beta_1(f(x)) = \gamma^{-1}(\beta(f(x))) = \gamma^{-1}(\beta(y)) = \beta_1(y)$. In view of the symmetric relation between (α, β) and (α_1, β_1) , the converse propositions are also true: if $\beta_1\alpha_1\beta_1 = \beta_1$, then $\beta\alpha\beta = \beta$, etc. It can also be shown similarly that β is a left or right inverse to α if and only if the same relation holds between β_1 and α_1 .

From these considerations the relation R is equivalent to the relation R_1 , in which α and β are replaced by α_1 and β_1 . Since $f'(x_1) = \alpha_1$, the conditions of the lemma are valid, and there is a function $g: B_1 \rightarrow A_1$ such that the pairs $(g(y), y)$ are the points of R_1 , and so of R , near (x_1, y_1) . For the point y_2 , of (ii), the lemma asserts $g'(y_2) = \gamma_1^{-1}\beta_1$, where $\gamma_1 = 1 + \beta(\alpha_2 - \alpha_1)$. But we have already shown that $\gamma_1^{-1}\beta_1 = [1 + \beta(\alpha_2 - \alpha)]^{-1}\beta$. This proves the theorem.

3. Other relations. In [1], the relations $gf\beta = \beta$, $g\alpha\beta = g$, and $gfg = g$ were discussed as local identities under restrictive conditions on the reference points. We can discuss them more generally in the light of the related function theorem above.

For any y , if $\beta(y) \in A$, then $(\beta(y), f(\beta(y))) \in R$, by easy calculation. If $f'(\beta(y_0)) = \alpha_1$, for one such y_0 and $1 + \beta(\alpha_1 - \alpha)$ has a bounded inverse, then the theorem gives a unique $g: B_1 \rightarrow A_1$, where A_1 is a neighborhood of $\beta(y_0)$ and B_1 is a neighborhood of $f(\beta(y_0))$, such that the pairs $(g(y), y)$ are in R . But for y near y_0 , the pairs $(\beta(y), f(\beta(y)))$ are points of R near $(\beta(y_0), f(\beta(y_0)))$, so by uniqueness of g , $\beta(y) = g(f(\beta(y)))$.

Similarly, if $(x, y) \in R$, then $(x, \alpha(\beta(y))) \in R$. If $(x_1, y_1) \in R$ and $f'(x_1) = \alpha_1$, satisfying the condition of the theorem, the theorem gives functions g and \bar{g} giving rise to pairs in R near the pairs (x_1, y_1) and $(x_1, \alpha(\beta(y_1)))$, respectively. Now if y is near y_1 , $(g(y), y) \in R$, so $(g(y), \alpha(\beta(y))) \in R$, and by uniqueness of \bar{g} , $g(y) = \bar{g}(\alpha(\beta(y)))$, for y near y_1 .

Finally, if $(x, y) \in R$, then $(x, f(x)) \in R$. For such a point (x_1, y_1) , if $f'(x_1) = \alpha_1$ is suitably well behaved, functions g and \bar{g} exist determining points of R near (x_1, y_1) and $(x_1, f(x_1))$, respectively. For y near y_1 , $(g(y), y) \in R$, so $(g(y), f(g(y))) \in R$, the latter point being near to $(x_1, f(x_1))$, and so by uniqueness of \bar{g} , $g(y) = \bar{g}(f(g(y)))$, for y near y_1 .

REFERENCE

1. E. B. Leach, *A note on inverse function theorems*, Proc. Amer. Math. Soc. **12** (1961), 694-697.

CASE INSTITUTE OF TECHNOLOGY