ON A RELATED FUNCTION THEOREM

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1. In a previous note [1], the strong differential, a variant of the classical Fréchet differential, was defined. Strong differentiability at a point seems to be a good smoothness condition for related function theorems, being stronger than the insufficient condition of Fréchet differentiability at the point and weaker than Fréchet differentiability in a neighborhood of the point, together with continuity of the differential at the point.

In this note we state as a lemma a slight generalization of the theorem of [1]. Algebraic manipulation of the relations involved then enables us to extend the range over which the conclusion of the lemma is valid.

2. The following definition is given in [1].

DEFINITION. Let A and B be open subsets of Banach spaces U and V, respectively, and let $f: A \to B$ be a function. We say that f has *strong differential* α at a point $x_0 \in A$, if $\alpha: U \to V$ is a bounded linear transformation and for every $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x'') - f(x') - \alpha(x'' - x')| \leqq \epsilon |x'' - x'|, \tag{1}$$

whenever $|x' - x_0| < \delta$ and $|x'' - x_0| < \delta$.

When f has a strong differential α at x_0 we shall write $f'(x_0) = \alpha$. The following lemma is the basic analytical tool of our discussion. (Throughout the rest of this note f will denote a fixed function relating open subsets A and B of Banach spaces U and V.)

LEMMA. *Let $\alpha: U \to V$ and $\beta: V \to U$ be bounded linear transformations such that $\beta\alpha\beta = \beta$. Let $x_0 \in A$ and $y_0 \in V$ satisfy $\beta(f(x_0)) = \beta(y_0)$ and $\beta(\alpha(x_0)) = x_0$. Finally, suppose that $f'(x_0) = \alpha$. Then there are neighborhoods A_0 of x_0 and B_0 of y_0 (with $A_0 \subset A$) such that:*

(i) *There is a unique function $g: B_0 \to A_0$ satisfying $\beta(\alpha(g(y))) = g(y)$ and $\beta(f(g(y))) = \beta(y)$, for all $y \in B_0$.*

(ii) *g is continuous, $g(y_0) = x_0$, and if for any $y_1 \in B_0$, $f'(g(y_1)) = \alpha_1$, then $g'(y_1) = \gamma^{-1}\beta$, where $\gamma = 1 + \beta(\alpha_1 - \alpha)$.*

The neighborhoods A_0 and B_0 are described in terms of an $\epsilon > 0$, chosen so that $\epsilon|\beta| < 1/2$, and a $\delta > 0$, chosen so that δ and ϵ satisfy the condition of differentiability of f at x_0 and so that the sphere of

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radius δ about x_0 is in A. Then A_0 is the sphere of radius δ about x_0 and B_0 is the sphere of radius $\delta/2|\beta|$ about y_0. The function g is the uniform limit of a sequence $\{g_n\}$ of functions defined recursively by:

$$(2) \qquad g_0(y) = x_0; \qquad g_{n+1}(y) = g_n(y) + \beta(y - f(g_n(y))), \quad \text{if } n \geqq 0.$$

The entire proof corresponds exactly with the proof of the theorem of [1], and will be omitted.

Let R be the relation consisting of pairs (x, y), such that $x \in A$, $y \in V$, $\beta(\alpha(x)) = x$ and $\beta(f(x)) = \beta(y)$. The lemma asserts that near a point (x_0, y_0) of R, points of R consist exactly of the pairs $(g(y), y)$, provided $f'(x_0) = \alpha$. The next theorem extends the range over which this conclusion is valid.

THEOREM. *Let α: $U \rightarrow V$ and β: $V \rightarrow U$ be bounded linear transformations such that $\beta\alpha\beta = \beta$. Let $x_1 \in A$ and $y_1 \in V$ be points satisfying $\beta(\alpha(x_1)) = x_1$ and $\beta(f(x_1)) = \beta(y_1)$. Finally, assume $f'(x_1) = \alpha_1$, a transformation such that $\gamma = 1 + \beta(\alpha_1 - \alpha)$ has a bounded inverse. Then there is a pair of neighborhoods A_1 of x_1 and B_1 of y_1 such that:*

(i) *There is a unique function g: $B_1 \rightarrow A_1$ satisfying $\beta(\alpha(g(y))) = g(y)$ and $\beta(f(g(y))) = \beta(y)$, for all $y \in B_1$.*

(ii) *g is continuous, $g(y_1) = x_1$ and if for any $y_2 \in B_1$, $f'(g(y_2)) = \alpha_2$, then $g'(y_2) = [1 + \beta(\alpha_2 - \alpha)]^{-1}\beta$.*

PROOF. For any bounded linear transformation α_2: $U \rightarrow V$, let $\beta_1 = \gamma^{-1}\beta$ and $\gamma_1 = 1 + \beta_1(\alpha_2 - \alpha_1)$. Then γ_1 has a bounded inverse if and only if $\gamma\gamma_1$ has a bounded inverse. In this case, let $\beta_2 = \gamma_1^{-1}\beta_1$. Then $\beta_2 = \gamma_1^{-1}\gamma^{-1}\beta = (\gamma\gamma_1)^{-1}\beta$, where $\gamma\gamma_1 = \gamma[1 + \gamma^{-1}\beta(\alpha_2 - \alpha_1)] = \gamma + \beta(\alpha_2 - \alpha_1) = 1 + \beta(\alpha_1 - \alpha) + \beta(\alpha_2 - \alpha_1) = 1 + \beta(\alpha_2 - \alpha)$. In particular, if $\alpha_2 = \alpha$, $\beta_2 = \beta$, and so the pairs (α, β) and (α_1, β_1) are symmetrically related.

Further, an identity of the type we are considering involving α and β may be replaced by the corresponding identity about α_1 and β_1. To see this, first note that because $\beta\alpha\beta = \beta$, then $\beta\alpha\gamma = \beta\alpha + \beta\alpha\beta(\alpha_1 - \alpha) = \beta\alpha_1$, and composing with γ^{-1} on the right we obtain:

$$(3) \qquad \beta\alpha = \beta\alpha_1\gamma^{-1}.$$

Also, $\gamma\beta = \beta + \beta(\alpha_1 - \alpha)\beta = \beta\alpha_1\beta$, and composing with γ^{-1} on the left we obtain:

$$(4) \qquad \beta = \gamma^{-1}\beta\alpha_1\beta.$$

Now $\beta_1\alpha_1\beta_1 = \gamma^{-1}\beta\alpha_1\gamma^{-1}\beta = \gamma^{-1}\beta\alpha\beta = \gamma^{-1}\beta = \beta_1$, using (3) at the second step. For any x, if $\beta(\alpha(x)) = x$, then $\beta_1(\alpha_1(x)) = \gamma^{-1}(\beta(\alpha_1(x))) = \gamma^{-1}(\beta(\alpha_1(\beta(\alpha(x))))) = \beta(\alpha(x)) = x$, using (4) at the third step. Finally, if $\beta(f(x))=\beta(y)$, for a pair (x, y), then $\beta_1(f(x))=\gamma^{-1}(\beta(f(x)))$ $=\gamma^{-1}(\beta(y))=\beta_1(y)$. In view of the symmetric relation between (α, β) and (α_1, β_1), the converse propositions are also true: if $\beta_1\alpha_1\beta_1=\beta_1$, then $\beta\alpha\beta=\beta$, etc. It can also be shown similarly that β is a left or right inverse to α if and only if the same relation holds between β_1 and α_1.

From these considerations the relation R is equivalent to the relation R_1, in which α and β are replaced by α_1 and β_1. Since $f'(x_1)=\alpha_1$, the conditions of the lemma are valid, and there is a function $g: B_1 \rightarrow A_1$ such that the pairs $(g(y), y)$ are the points of R_1, and so of R, near (x_1, y_1). For the point y_2, of (ii), the lemma asserts $g'(y_2)$ $=\gamma_1^{-1}\beta_1$, where $\gamma_1=1+\beta(\alpha_2-\alpha_1)$. But we have already shown that $\gamma_1^{-1}\beta_1=[1+\beta(\alpha_2-\alpha)]^{-1}\beta$. This proves the theorem.

3. **Other relations.** In [1], the relations $gf\beta=\beta$, $g\alpha\beta=g$, and $gfg=g$ were discussed as local identities under restrictive conditions on the reference points. We can discuss them more generally in the light of the related function theorem above.

For any y, if $\beta(y)\in A$, then $(\beta(y), f(\beta(y)))\in R$, by easy calculation. If $f'(\beta(y_0))=\alpha_1$, for one such y_0 and $1+\beta(\alpha_1-\alpha)$ has a bounded inverse, then the theorem gives a unique $g: B_1\rightarrow A_1$, where A_1 is a neighborhood of $\beta(y_0)$ and B_1 is a neighborhood of $f(\beta(y_0))$, such that the pairs $(g(y), y)$ are in R. But for y near y_0, the pairs $(\beta(y), f(\beta(y)))$ are points of R near $(\beta(y_0), f(\beta(y_0)))$, so by uniqueness of g, $\beta(y)$ $=g(f(\beta(y)))$.

Similarly, if $(x, y)\in R$, then $(x, \alpha(\beta(y)))\in R$. If $(x_1, y_1)\in R$ and $f'(x_1)=\alpha_1$, satisfying the condition of the theorem, the theorem gives functions g and $\bar{g}$ giving rise to pairs in R near the pairs (x_1, y_1) and $(x_1, \alpha(\beta(y_1)))$, respectively. Now if y is near y_1, $(g(y), y)\in R$, so $(g(y), \alpha(\beta(y)))\in R$, and by uniqueness of $\bar{g}$, $g(y)=\bar{g}(\alpha(\beta(y)))$, for y near y_1.

Finally, if $(x, y)\in R$, then $(x, f(x))\in R$. For such a point (x_1, y_1), if $f'(x_1)=\alpha_1$ is suitably well behaved, functions g and $\bar{g}$ exist determining points of R near (x_1, y_1) and $(x_1, f(x_1))$, respectively. For y near y_1, $(g(y), y)\in R$, so $(g(y), f(g(y)))\in R$, the latter point being near to $(x_1, f(x_1))$, and so by uniqueness of $\bar{g}$, $g(y)=\bar{g}(f(g(y)))$, for y near y_1.

Reference

1. E. B. Leach, *A note on inverse function theorems*, Proc. Amer. Math. Soc. **12** (1961), 694–697.

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