Σ-SYMMETRIC LOCALLY CONVEX SPACES

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In [1] it is shown that barrelledness and quasi-barrelledness are merely the two extreme examples of a property, called Σ-symmetry, which may be possessed by a locally convex Hausdorff topological vector space. The object of this note is to show how recent characterisations [2; 3] of barrelled and quasi-barrelled spaces may be subsumed under characterisations of Σ-symmetric spaces, and to exhibit some properties of these spaces. First we need some definitions and simple results.

1. Let E be a locally convex Hausdorff topological vector space (abbreviated to LCS in what follows), and let Σ be a class of bounded subsets of E whose union is E. Let E' denote the topological dual of E, and let E'_Σ be the set E' endowed with the topological of uniform convergence on the members of Σ.

DEFINITION 1. A subset of E is said to be *Σ-bornivorous* if it absorbs every member of Σ.

DEFINITION 2. We say that E is *Σ-symmetric* if any of the following equivalent conditions hold:

(a) Every Σ-bornivorous barrel in E is a neighbourhood of zero.

(b) Every bounded subset of E'_Σ is equicontinuous.

(c) The topology induced on E by the strong dual of E'_Σ is the original topology of E.

The equivalence of these conditions was proved in [1]. If $\Sigma_1 \subset \Sigma_2$ it is easy to see that Σ_1-symmetry implies Σ_2-symmetry; the strongest restriction on E is obtained by taking for Σ the class s of all subsets of E consisting of a single point, and then Σ-symmetry is simply the property of being barrelled. If Σ is the class b of all bounded subsets of E we have the weakest Σ-symmetric property, which is that of being quasi-barrelled. Whatever the choice of Σ, the topology of a Σ-symmetric space is the Mackey topology [1].

DEFINITION 3. E is said to be *Σ-bornological* if every convex Σ-bornivorous subset of E is a neighbourhood of zero.

It follows easily that E is Σ-bornological if and only if every linear map of E into an LCS F which takes members of Σ into bounded sets in F is continuous. For all choices of Σ, a Σ-bornological space is bornological.

Given any LCS E and any class Σ of bounded sets whose union is

Received by the editors August 13, 1962.

E, we may define on E a new topology $\mathfrak{I}$ by taking as a fundamental system of neighbourhoods of zero the class of all convex circled Σ-bornivorous subsets of E. $\mathfrak{I}$ is the finest locally convex topology for which the members of Σ remain bounded, and the space F obtained by endowing the point set E with $\mathfrak{I}$ is Σ-bornological. F is called the Σ-bornological space *associated* with E. We note that when $\Sigma = s$, $\mathfrak{I}$ becomes the finest locally convex topology on E, and when $\Sigma = b$, $\mathfrak{I}$ is the associated bornological topology [**4**, Chapitre 3, §2, Exercice 13].

Definition 4. Let E_1, E_2 be LCS. A linear mapping u from E_1 onto E_2 is said to be *almost open* if for every neighbourhood U of zero in E_1, the closure of $u(U)$ in E_2 is a neighbourhood of zero for the Mackey topology on E_2.

Definition 5. A linear subspace Q of the dual E' of an LCS E is said to be *almost closed* if for every neighbourhood U of zero in E, $U^0 \cap Q$ is weakly closed in E', where U^0 denotes the polar of U.

2. The theorems in this section give characterisations of Σ-symmetric spaces. The first result includes Theorems 2.3 and 2.4 of [**2**] as special cases.

Theorem 1. *Let E be an LCS and Σ a class of bounded subsets of E whose union is E. Let F be the Σ-bornological space associated with E. Then E is Σ-symmetric if and only if the topology of E is the Mackey topology and either of the following conditions holds*:

(a) *The identity map from F onto E is almost open.*

(b) *E' is almost closed in F'.*

Proof. This follows the pattern of the corresponding proof in [**2**]; we give some details for convenience. To prove that Σ-symmetry implies (b), all we need show, since F has the Mackey topology, is that $E' \cap K$ is weakly closed in F' for every weakly compact convex circled subset K of F'. Since K is an equicontinuous subset of F' it is bounded in F'_Σ, so that $E' \cap K$ is bounded in E'_Σ, E'_Σ being plainly a topological subspace of F'_Σ. Since E is Σ-symmetric, $E' \cap K$ is thus an equicontinuous subset of E', and is hence relatively weakly compact in E'. Actually $E' \cap K$ is weakly compact in E', since E'_s is a topological subspace of F'_s, and K is weakly closed in F'. It follows that $K \cap E'$ is weakly closed in F'.

Condition (a) and the Mackey condition imply Σ-symmetry, since if U is a Σ-bornivorous barrel in E it is the closure in E of a neighbourhood of zero in F, so that by (a), U is a neighbourhood of zero in E.

This completes the proof of the theorem, since Pták [**5**] has shown that (a) and (b) are equivalent.

The next theorem is a characterisation in terms of the closed graph theorem, and contains Theorems 2.2 and 3.1 of [3] as particular cases.

THEOREM 2. *Let E, Σ be as in Theorem* 1. *Then E is Σ-symmetric if and only if for every Banach space F the following is true:*

If u is any linear map from E into F such that

(a) *u takes members of Σ into bounded sets in F;*

(b) *the graph of u is closed;*

then u is continuous.

The proof is an obvious modification of that given in [3].

3. We conclude by indicating various general properties of Σ-symmetric and Σ-bornological spaces.

THEOREM 3. (a) *If E is Σ-symmetric, E'_Σ is quasi-complete.* (b) *Let Σ be a class of convex, circled, closed bounded sets whose union is E. Then if E is Σ-bornological, E'_Σ is complete.*

PROOF. (a) Let B be a bounded closed subset of E'_Σ. Since E is Σ-symmetric, B is equicontinuous and is therefore complete [4, Chapitre 3, §3, Théorème 4].

(b) The completion of E'_Σ is the set of all linear functionals on E whose restriction to each member of Σ is continuous [4, Chapitre 4, §3, Exercice 3]. Such functionals are bounded on the members of Σ, and since E is Σ-bornological they are continuous on E. Hence E'_Σ coincides with its completion.

Specialisations of this theorem give, for example, the familiar results that the dual of a quasi-barrelled space is strongly quasi-complete, and that the strong dual of a bornological space is complete.

THEOREM 4. *Let $(E_i)_{i\in I}$ be any family of LCS, and for each $i\in I$ let Σ_i be a class of convex circled bounded subsets of E_i whose union is E_i. Let Σ be the class of all subsets of $E=\prod_{i\in I} E_i$ of the form $\prod_{i\in I} S_i$, $S_i\in\Sigma_i$. Then if for all $i\in I$, E_i is Σ_i-symmetric, E is Σ-symmetric when endowed with the usual product topology.*

PROOF. Let B be any bounded subset of E'_Σ. We need to prove that B is equicontinuous. The topology of E'_Σ is simply the topological direct sum of the topologies of the $(E_i)'_{\Sigma_i}$ [6, Chapitre 4, §1, Proposition 7]. It follows [6, Chapitre 4, §1, Proposition 5] that B is contained and bounded in the direct sum of a finite number of the $(E_i)'_{\Sigma_i}$, i.e. $B\subset\sum_{i\in H}(E_i)'_{\Sigma_i}$, H finite. Hence for each $i\in H$ the projection of B into $(E_i)'_{\Sigma_i}$ is bounded and so is equicontinuous, since E_i is Σ_i-symmetric. B is thus an equicontinuous subset of E' [6, Chapitre 2, §15, Proposition 22, Corollaire 1], which completes the proof.

By choosing the Σ_i suitably we can obtain as special cases of this theorem the well-known results that the product of any family of barrelled (resp. quasi-barrelled) spaces is barrelled (resp. quasi-barrelled.)

THEOREM 5. *Let E be Σ-symmetric and let M be a closed subspace of E. Denote by ϕ the canonical mapping $E \to E/M$, and put $\Sigma_1 = \{\phi(S): S \in \Sigma\}$. Then E/M is Σ_1-symmetric.*

The proof is obvious.

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