A NOTE ON ABSOLUTE $G_δ$-SPACES

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A set $X$, which is a $G_δ$ in a compact space is called an absolute $G_δ$-space (simply-absolute $G_δ$) or a topologically complete space. It was noted by Knaster that there exist two classes $\mathfrak{A}$ and $\mathfrak{B}$ of such spaces, where by definition:

$X \in \mathfrak{A}$ if $X$ is an absolute $G_δ$ and there exists a homeomorphism $h: X \rightarrow Y$ of $X$ into a compact space $Y$, such that the image $h(X)$ of $X$ can be written in the form:

$\bigcap_{i=1}^{\infty} G_i$ with $\dim Fr(G_i) < \dim X$ and $G_i$ open in $Y$,

$X \in \mathfrak{B}$ if $X$ is an absolute $G_δ$ and $X \in \mathfrak{B}$. Knaster also showed that the set $N \times D$, where $N$ is the set of irrational numbers of the interval $D = [0, 1]$, belongs to $\mathfrak{B}$. In answer to one of his questions, it was proved by Lelek that every set of the form $N \times Z$, where $Z$ is an arbitrary finite-dimensional compact set, belongs to $\mathfrak{B}$. Lelek posed also the following:

PROBLEM. Does there exist for every metric, separable and topologically complete, finite-dimensional space $X \in \mathfrak{B}$, with $\dim X > 0$, a compact space $Z$, with $\dim Z > 0$ such that the set $N \times Z$ has a topological image in $X$?

The aim of this paper is to give a negative answer to this problem. This will be done by the following:

Example of a set $X \in \mathfrak{B}$, with $\dim X = 1$, which does not contain a topological image of any set of the form $N \times Z$ with $Z$ compact and $\dim Z > 0$.

Let namely $D_n$ be the closed unit interval joining the points $p_n = (1/n, 0)$ and $q_n = (1/n, 1)$ in the $(x, y)$-plane, $n = 1, 2, \ldots$ (i.e., $D_n = \{ (x, y); x = 1/n, 0 \leq y \leq 1 \}$) and let $p = (0, 0)$ and $q = (0, 1)$.

We put $X = \bigcup_{n=1}^{\infty} D_n \cup \{ \bar{p} \} \cup \{ q \}$. Evidently $\dim X = 1$, and it suffices to show that the set $X$ has the following properties:

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1 Only metric, separable spaces are considered.
2 See [1, p. 264].
3 $\dim X$ denotes the dimension of $X$; $Fr(X)$ is the boundary of $X$.
4 See [1, pp. 263–264].
5 See [3, p. 34].
6 See [3, p. 34]. The author learned recently that this problem has also been solved, in an entirely different way, by A. Lelek (unpublished to date).
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(a) $X$ is an absolute $G$,\textsuperscript{7}

(b) $X \subseteq \mathcal{F}$ and

c) given a set $T$ of the form $T = N \times Z$ with $Z$ compact and $\dim Z > 0$, there does not exist a homeomorphism of $T$ into $X$.

To show (a), note that the closure $\overline{X}$ of $X$ in the $(x, y)$-plane equals:
$$\overline{X} = \bigcup_{n=1}^{\infty} D_n \cup D_0,$$
where $D_0 = \{(x, y) : x = 0, 0 \leq y \leq 1\}$ and $\overline{X}$ is a compact space. It differs from $X$ by the open interval $D_0 - \{(p)^J(q)\}$, which is an $F$ in $X$. Therefore (a) holds.

To show (b), we shall prove that the assumption $X \subseteq \mathcal{F}$ leads to a contradiction.

Suppose, that $X \subseteq \mathcal{F}$. Then:

1) There exists a homeomorphism $h : X \to Y$ of $X$ into a compact space $Y$, such that $\dim \{Y - h(X)\} < \dim X = 1$.\textsuperscript{8}

Since the intervals $D_n$ are disjoint, the sets $h(D_n); n = 1, 2, \ldots$ form a sequence of disjoint continua (even arcs) in the compact space $Y$. Thus, there exists a subsequence $\{k\}$ of natural numbers, such that the continua $h(D_k)$ converge to a continuum $E = \lim_{k \to \infty} h(D_k)$.\textsuperscript{9}

Now, it is easily seen that

$$(b_1) \quad \text{the diameter } \delta(E) > 0.$$ 

Indeed, if $E$ were to reduce to a point $\hat{p}$, there would be, for the endpoints $p_k$ and $q_k$ of $D_k$: $p_k \to \hat{p}$, $h(p_k) \to \hat{p} = h(p)$ and $q_k \to \hat{q}$, $h(q_k) \to \hat{q} = h(q)$ which is impossible, since $h$ is a one-to-one mapping.

We also have

$$(b_2) \quad E \cap h(D_n) = 0 \quad \text{for every } n = 1, 2, \ldots$$

since otherwise there would exist a number $n_0$, a point $r \in D_{n_0}$, a subsequence $\{j\}$ of $\{k\}$ and points $r_j \in D_j$ such that $\lim_{j \to \infty} r_j = r$ which is impossible by the definition of the intervals $D_n$. (No interval $D_n$ contains a limit point of a sequence of points belonging to intervals $D_n$ for $n \neq n_0$.)

By (b$_1$), $E$ is a continuum containing more than one point and therefore $\dim E \geq 1$. But by (b$_2$) we have $E \subseteq Y - h(\bigcup_{n=1}^{\infty} D_n)$. Hence by $h(X) = h(\bigcup_{n=1}^{\infty} D_n) \cup (h(p)) \cup (h(q))$ we have $\dim \{Y - h(X)\} \geq 1$ which contradicts (1).

Thus (b) holds. It remains to show (c). For this purpose suppose, to the contrary, that there would exist a compact set $Z$ with $\dim Z > 0$

\textsuperscript{7} It is easily seen that $X$ is also an absolute $F$, i.e. an $F$ in a compact space.

\textsuperscript{8} This is a trivial consequence of [3, p. 31, Theorem 1]. See also the remark at the end of the present paper.

\textsuperscript{9} This follows from [2, p. 110, Theorem 4]. It can also be derived from [5, p. 11, (9, 11)].
such that the set $T = N \times Z$ has a topological image $f(T)$ in $X$. Since \( \dim Z > 0 \), the compact set $Z$ contains a continuum $C$ which does not reduce to one point.\(^{10}\) Therefore the set $T = N \times Z$ would contain the set $N \times C$ which consists of $2^{\aleph_0}$ disjoint continua $C_i$ and we could write $N \times C = \bigcup_{i \in \mathbb{N}} C_i$. The image $f(C_i)$ of every $C_i$ would be a continuum contained in $X$.\(^{11}\) But $X$ is a union of a denumerable sequence of closed sets. Hence, by a theorem of Sierpinski\(^{12}\) the set $f(C_i)$ has to be contained in one and only one, interval $D_n = D_{n(t)}$ $n = 1, 2, \ldots$. Thus $f(C_i)$ would be, for every $\xi$, a closed interval contained in an interval $D_n(\xi)$. Now for $\xi' \neq \xi''$ the intervals $f(C_{\xi'})$ and $f(C_{\xi''})$ would be disjoint and therefore, there would exist a family of power $2^{\aleph_0}$ of disjoint intervals contained in the set $X$, which is impossible (since $X$ is a union of a countable family of intervals and two points). Therefore (c) also holds.

Remark. As noted in footnote 8, the proof of (1) is a consequence of Theorem 1, p. 31 of \[3\]. This theorem concerns finite-dimensional spaces. Now it is easily seen that (1) follows also from the fact that

(2) If $X \in \mathfrak{A}$, then there exists a compact space $Y$ and a homeomorphism $h: X \rightarrow Y$ such that $h(X) = \bigcap_{i=1}^{\infty} G_i$ where $G_i$ are open in $Y$ and $Y - h(X) = \bigcup_{i=1}^{\infty} \text{Fr}(G_i)$ with $\dim \text{Fr}(G_i) < \dim X$ $i = 1, 2, \ldots$.

Indeed, if $X \in \mathfrak{A}$ then $X$ can be represented in the form $(1^o)$. Taking the closure $\text{cl}(h(X))$ of $h(X)$ in $Y$ and denoting this “new” set $\text{cl}(h(X))$ by $Y$ and the “new” sets $G_i \cap \text{cl}(h(X))$ by $G_i$, it is easy to verify that (2) holds without any assumption of finite-dimensionality of $X$.

References


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\(^{10}\) See [2, p. 130]; also [4, p. 278].

\(^{11}\) Evidently $f(C_i)$ contains more than one point.

\(^{12}\) See [2, p. 113].

\(^{18}\) The proof is analogous to that of the necessity of Theorem 1, p. 31 of [3].