A NOTE ON ABSOLUTE G_δ-SPACES

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A set X, which is a G_δ in a compact space[1] is called an absolute G_δ-space (simply-absolute G_δ) or a topologically complete space. It was noted by Knaster[2] that there exist two classes $\mathfrak{A}$ and $\mathfrak{B}$ of such spaces, where by definition:

$X \in \mathfrak{A}$ if X is an absolute G_δ and there exists a homeomorphism $h: X \to Y$ of X into a compact space Y, such that the image $h(X)$ of X can be written in the form:

$$(1^\circ) \qquad h(X) = \bigcap_{i=1}^{\infty} G_i, \text{ with } \dim \operatorname{Fr}(G_i) < \dim X \text{ and } G_i \text{ open in } Y, \quad i = 1, 2, \cdots.$$ [3]

$X \in \mathfrak{B}$ if X is an absolute G_δ and $X \notin \mathfrak{A}$. Knaster also showed,[4] i.a., that the set $N \times D$, where N is the set of irrational numbers of the interval $D = [0, 1]$, belongs to $\mathfrak{B}$. In answer to one of his questions, it was proved by Lelek[5] that every set of the form $N \times Z$, where Z is an arbitrary finite-dimensional compact set, belongs to $\mathfrak{B}$. Lelek posed also the following:

PROBLEM.[6] Does there exist for every metric, separable and topologically complete, finite-dimensional space $X \in \mathfrak{B}$, with $\dim X > 0$, a compact space Z, with $\dim Z > 0$ such that the set $N \times Z$ has a topological image in X?

The aim of this paper is to give a negative answer to this problem. This will be done by the following:

Example of a set $X \in \mathfrak{B}$, with $\dim X = 1$, which does not contain a topological image of any set of the form $N \times Z$ with Z compact and $\dim Z > 0$.

Let namely D_n be the closed unit interval joining the points $p_n = (1/n, 0)$ and $q_n = (1/n, 1)$ in the (x, y)-plane, $n = 1, 2, \cdots$ (i.e., $D_n = \{(x, y); x = 1/n, 0 \leqq y \leqq 1\}$) and let $p = (0, 0)$ and $q = (0, 1)$.

We put $X = \bigcup_{n=1}^{\infty} D_n \cup (p) \cup (q)$. Evidently $\dim X = 1$, and it suffices to show that the set X has the following properties:

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[1] Only metric, separable spaces are considered.

[2] See [1, p. 264].

[3] $\dim X$ denotes the dimension of X; $\operatorname{Fr}(X)$ is the boundary of X.

[4] See [1, pp. 263–264].

[5] See [3, p. 34].

[6] See [3, p. 34]. The author learned recently that this problem has also been solved, in an entirely different way, by A. Lelek (unpublished to date).

(a) X is an absolute G_δ,[7]

(b) $X \in \mathfrak{B}$ and

(c) given a set T of the form $T = N \times Z$ with Z compact and $\dim Z > 0$, there does not exist a homeomorphism of T into X.

To show (a), note that the closure $\overline{X}$ of X in the (x, y)-plane equals: $\overline{X} = \bigcup_{n=1}^{\infty} D_n \cup D_0$, where $D_0 = \{(x, y); x = 0, 0 \leqq y \leqq 1\}$ and $\overline{X}$ is a compact space. It differs from X by the open interval $D_0 - [(p) \cup (q)]$, which is an F_σ in $\overline{X}$. Therefore (a) holds.

To show (b), we shall prove that the assumption $X \in \mathfrak{A}$ leads to a contradiction.

Suppose, that $X \in \mathfrak{A}$. Then:

(1) There exists a homeomorphism $h: X \to Y$ of X into a compact space Y, such that $\dim [Y - h(X)] < \dim X = 1$.[8]

Since the intervals D_n are disjoint, the sets $h(D_n)$; $n = 1, 2, \cdots$ form a sequence of disjoint continua (even arcs) in the compact space Y. Thus, there exists a subsequence $\{k\}$ of natural numbers, such that the continua $h(D_k)$ converge to a continuum $E = \lim_{k \to \infty} h(D_k)$.[9]

Now, it is easily seen that

(b₁) $$\text{the diameter } \delta(E) > 0.$$

Indeed, if E were to reduce to a point $\bar{p}$, there would be, for the endpoints p_k and q_k of D_k: $p_k \to p$, $h(p_k) \to \bar{p} = h(p)$ and $q_k \to q$, $h(q_k) \to \bar{p} = h(q)$ which is impossible, since h is a one-to-one mapping.

We also have

(b₂) $$E \cap h(D_n) = 0 \qquad \text{for every } n = 1, 2, \cdots$$

since otherwise there would exist a number n_0, a point $r \in D_{n_0}$, a subsequence $\{j\}$ of $\{k\}$ and points $r_j \in D_j$ such that $\lim_{j \to \infty} r_j = r$ which is impossible by the definition of the intervals D_n. (No interval D_{n_0} contains a limit point of a sequence of points belonging to intervals D_n for $n \neq n_0$.)

By (b₁), E is a continuum containing more than one point and therefore $\dim E \geqq 1$. But by (b₂) we have $E \subset Y - h(\bigcup_{n=1}^{\infty} D_n)$. Hence by $h(X) = h(\bigcup_{n=1}^{\infty} D_n) \cup (h(p)) \cup (h(q))$ we have $\dim [Y - h(X)] \geqq 1$ which contradicts (1).

Thus (b) holds. It remains to show (c). For this purpose suppose, to the contrary, that there would exist a compact set Z with $\dim Z > 0$

[7] It is easily seen that X is also an absolute F_σ, i.e. an F_σ in a compact space.

[8] This is a trivial consequence of [3, p. 31, Theorem 1]. See also the remark at the end of the present paper.

[9] This follows from [2, p. 110, Theorem 4]. It can also be derived from [5, p. 11, (9, 11)].

such that the set $T=N\times Z$ has a topological image $f(T)$ in X. Since dim $Z>0$, the compact set Z contains a continuum C which does not reduce to one point.[10] Therefore the set $T=N\times Z$ would contain the set $N\times C$ which consists of $2^{\aleph_0}$ disjoint continua C_ξ and we could write $N\times C=\bigcup_{\xi\in N} C_\xi$. The image $f(C_\xi)$ of every C_ξ would be a continuum contained in X.[11] But X is a union of a denumerable sequence of closed sets. Hence, by a theorem of Sierpinski[12] the set $f(C_\xi)$ has to be contained in one and only one, interval $D_n=D_{n(\xi)}$ $n=1, 2, \cdots$. Thus $f(C_\xi)$ would be, for every ξ, a closed interval contained in an interval $D_{n(\xi)}$. Now for $\xi'\neq\xi''$ the intervals $f(C_{\xi'})$ and $f(C_{\xi''})$ would be disjoint and therefore, there would exist a family of power $2^{\aleph_0}$ of disjoint intervals contained in the set X, which is impossible (since X is a union of a countable family of intervals and two points). Therefore (c) also holds.

REMARK. As noted in footnote 8, the proof of (1) is a consequence of Theorem 1, p. 31 of [3]. This theorem concerns finite-dimensional spaces. Now it is easily seen that (1) follows also from the fact that

(2) If $X\in\mathfrak{A}$, then there exists a compact space Y and a homeomorphism $h: X\to Y$ such that $h(X)=\bigcap_{i=1}^{\infty} G_i$ where G_i are open in Y and $Y-h(X)=\bigcup_{i=1}^{\infty}\mathrm{Fr}(G_i)$ with dim $\mathrm{Fr}(G_i)<\dim X$[13] $i=1, 2, \cdots$.

Indeed, if $X\in\mathfrak{A}$ then X can be represented in the form (1°). Taking the closure $\mathrm{cl}(h(X))$ of $h(X)$ in Y and denoting this "new" set $\mathrm{cl}(h(X))$ by Y and the "new" sets $G_i\cap \mathrm{cl}(h(X))$ by G_i, it is easy to verify that (2) holds without any assumption of finite-dimensionality of X.

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[10] See [2, p. 130]; also [4, p. 278].

[11] Evidently $f(C_\xi)$ contains more than one point.

[12] See [2, p. 113].

[13] The proof is analogous to that of the necessity of Theorem 1, p. 31 of [3].