REMARKS ON BALANCED INCOMPLETE BLOCK DESIGNS

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A *balanced incomplete block design* (BIBD) is a class of b subsets, or *blocks*, from a set of v elements with k elements in each block; each element is in r blocks and each pair of distinct elements is in λ blocks. We shall establish the following

THEOREM. *Let D be a BIBD with parameters* $(v, b, k, r, \lambda) = (2x+2, 4x+2, x+1, 2x+1, x)$, *where x is an even positive integer. Then* (i) *any two blocks of D have at least one common element*; (ii) *no two blocks of D are the same subset.*

PROOF. Assume the hypothesis and the falsity of either conclusion. Construct matrix A of $2x+2$ rows and $4x+4$ columns, with entries $+1$ and -1. The first column contains exclusively $+1$, and the second column -1. Set up one-to-one correspondences between rows of A and elements of D; between columns other than the first two of A and blocks of D. Enter $+1$ if the element is contained in the block and -1 otherwise. Then each row of A contains exactly $1+0+(2x+1)=2x+2$ entries $+1$, and hence $2x+2$ entries -1. Further, each pair of distinct rows contains two $+1$'s in exactly $1+0+x=x+1$ like columns. It follows that each pair of distinct rows of A has in like columns the ordered pairs $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$ each $x+1$ times. Select $x+1$ rows corresponding to the elements of a block which is either repeated in D or disjoint from another block of D. Let A_0 be the $x+1$ by $4x+4$ submatrix of A composed of these rows. Then $A_0A_0^T=(4x+4)I$, with the identity matrix of dimension $x+1$. A_0 has four columns each with all entries equal; these are the first two and those corresponding to the pair of special blocks of D. All pairs of unequal ± 1 entries in like columns of A_0 therefore occur in the other $4x$ columns. Each pair of distinct rows of A_0 contains $2x+2$ unlike entries in like columns; thus the total number of pairs of unequal entries within columns is $(x+1)x\cdot(2x+2)/2=(x+1)^2x$. Among the $4x$ columns the average number of unlike pairs is accordingly $(x+1)^2x/4x=(x+1)^2/4$. A partition of $x+1$ elements into two classes, with as many as $(x+1)^2/4$ pairs of elements not in the same class, is possible only if $x+1$ is even.

There exist numerous examples of BIBD with parameters as in the theorem and x an *odd* positive integer. (It is likely in fact that a design exists for each choice of x. This would be a corollary of Paley's

Received by the editors June 21, 1962.

very plausible conjecture that a so-called Hadamard matrix exists for order each multiple of four. Such a matrix has elements ± 1 and all inner products zero between distinct rows (see [1] and bibliography thereof). The converse implication might be false.) For odd x conclusion (i) of the theorem need not be true; an example is outlined for $x=3$. Form the symmetric and cyclic BIBD with parameters $(v, b, k, r, \lambda)=(15, 15, 7, 7, 3)$ determined by the difference set 0, 1, 2, 4, 5, 8, 10 (mod 15). Delete one block and all its elements where they occur in the other blocks, leaving a BIBD with parameters (8, 14, 4, 7, 3). It is easily verified that the latter BIBD has a pair (in fact seven pairs) of blocks without common element.

Whether conclusion (ii) holds for odd x appears less easy to decide; the above proof is not valid, but the author has constructed no counterexample. It is seldom if ever stated explicitly that all blocks of a BIBD must be distinct. Examples of BIBD are constructed easily with pairs of like blocks by choosing a quintuple of parameters for which a solution exists, then multiplying b, r, and λ by an integer greater than one; it is rather understood that b, r, and λ should not have a common prime divisor, for this bad property means in effect that a designed experiment would be doubled, tripled, etc., in size. Even for b, r, λ lacking a common divisor, there exist BIBDs with pairs of like blocks. An example is presented with parameters (10, 30, 3, 9, 2). The ten elements are designated by X, Y, Z; 0, 1, 2, 3, 4, 5, 6. The thirty triples are XYZ twice; and four classes of seven blocks each obtained by adding (mod 7) all constants to the digits in X01, Y02, Z04, and 124, leaving X, Y, Z fixed by the addition. Thus the second half of the theorem is not vacuous in content.

Reference

1. Leonard Baumert, S. W. Golomb, and Marshall Hall, Jr., *Discovery of an Hadamard matrix of order* 92, Bull. Amer. Math. Soc. **68** (1962), 237–238.

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