A SUPPLEMENT TO PARKER'S "REMARKS ON BALANCED INCOMPLETE BLOCK DESIGNS"[1]

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1. E. T. Parker proved [2] the following theorem.

Let D be a balanced incomplete block design (BIBD) with parameters $(v, b, k, r, \lambda) = (2x+2, 4x+2, x+1, 2x+1, x)$, *where x is a positive even integer. Then* (i) *any two blocks of D have at least one common element, and* (ii) *no two blocks of D are the same subset.*

Parker showed further that conclusion (i) need not hold when x is odd. He concludes his remarks saying, "Whether conclusion (ii) holds for odd x appears less easy to decide." It is the purpose of this note to show that conclusion (ii) holds also for x odd.

The result of this note is established using formulae of Bose and Bush [1] connecting parameters of orthogonal arrays $(\lambda s^2, k, s, 2)$:

(a) $\sum_{0 \leq i \leq k} n_i = \lambda s^2 - 1$,

(b) $\sum_{0 \leq i \leq k} i n_i = k(\lambda s - 1)$,

(c) $\sum_{0 \leq i \leq k} i(i-1) n_i = k(k-1)(\lambda - 1)$,

where n_i denotes the number of columns which have i coincidences with any chosen fixed column of the array, λ is the frequency of each ordered pair in like columns of each pair of distinct rows of the array, k is the number of rows, and s is the number of values the elements of the array can take on.

2. **On conclusion (ii) of Parker's Theorem.** Conclusion (ii) holds for any positive integer x.

PROOF. Parker shows that the existence of BIBD with parameters $(2x+2, 4x+2, x+1, 2x+1, x)$, x a positive integer, implies the existence of an orthogonal array with $s=2$, $\lambda = x+1$, $k = 2x+2$. If conclusion (ii) does not hold then this array would have to include at least two identical columns. This means that equations (a), (b), and (c) would have to hold with $n_{2\lambda} \geqq 1$. For $k = 2\lambda$, $s = 2$ equations (a), (b) and (c) reduce to

$$\sum_{0 \leq i \leq 2\lambda} n_i = 4\lambda - 1$$

$$\sum_{0 \leq i \leq 2\lambda} i n_i = 2\lambda(2\lambda - 1)$$

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$$\sum_{0 \leq i \leq 2\lambda} i(i-1)n_i = 2\lambda(2\lambda-1)(\lambda-1).$$

Hence, the average number of coincidences, say, $\bar{n} = 2\lambda(2\lambda-1)/(4\lambda-1)$. Furthermore $\sum_{0 \leq i \leq 2\lambda} (i-\bar{n})^2 n_i = 2\lambda^2(2\lambda-1)/(4\lambda-1)$. If $n_{2\lambda} = 1$ then the contribution of this term alone to the total sum of squares equals $16\lambda^4/(4\lambda-1)^2$ which is clearly impossible.

References

1. R. C. Bose and K. A. Bush, *Orthogonal arrays of strength two and three*, Ann. Math. Statist. 23 (1952), 508–524.

2. E. T. Parker, *Remarks on balanced incomplete block designs*, Proc. Amer. Math. Soc. 14 (1963), 729–730.

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