IMBEDDING AND IMMERSION OF REAL PROJECTIVE SPACES

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We will prove the following results on the imbeddability and immersibility of real n-dimensional projective space P^n in Euclidean space. All imbeddings and immersions are differentiable.

A. *If $n-1$ is a power of two, P^n cannot be imbedded in E^{2n-2}.*

B. *If $n>7$, P^n cannot be immersed in E^{n+2}.*

C.[1] *P^9 can be immersed in E^{15}.*

Since P^n can be imbedded in E^{2n-1} for n odd (see [4]), (A) solves the imbedding problem for P^n when $n-1$ is a power of two. The immersion problem for P^9 is solved by (C), since a consideration of normal Stiefel-Whitney classes shows P^9 cannot be immersed in E^{14}. The immersion problem for P^n, $n<9$, has already been solved (see 7.1 of [2]).

PROOF OF A. We will make use of the techniques introduced by Massey in [5] and [6]. This result is, in a sense, optimal for this method. Let M denote P^n. We will need to know that the normal Stiefel-Whitney classes of M, $\bar{w}_1$ and $\bar{w}_{n-3}$, are zero and nonzero, respectively, when $n-1$ is a power of two.

Suppose M is imbedded in E^{2n-2}. Let N be the normal $(n-3)$-sphere bundle and $p: N \to M$ the bundle projection. Recall the following information about the mod 2 cohomology of N (see [6] for details and references).

(i) There is a subring A of $H^*(N)$ with the following properties:

(a) A is closed under cohomology operations.

(b) $A^{2n-3}=0$.

(c) For every x in $H^r(N)$, $0<r<2n-3$, there are unique elements x_1 in $H^r(M)$ and y in A^r such that:

$$x = p^*(x_1) + y.$$

(ii) There is a unique element a in A^{n-3} with the following properties:

(a) For every x in $H^*(N)$, there are unique elements x_1, x_2 in $H^*(M)$ such that:

$$x = p^*(x_1) + a \cup p^*(x_2).$$

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[1] I have been informed by B. J. Sanderson that he has immersed P^n in E^{2n-3}, for n odd.

(b) In the formula of (a), if $x = Sq^r(a)$ then $x_2 = \bar{w}_r$, the rth normal Stiefel-Whitney class of M.

Let u be the generator of $H^1(M)$. By (i-c) we can define an element k in Z_2 by the formula:

$$a \cup p^*(u) + kp^*(u^{n-2}) \in A^{n-2}. \tag{1}$$

We now apply Sq^1 to (1). By (i-a), (ii-b) and $\bar{w}_1 = 0$, we obtain:

$$a \cup p^*(u^2) + kp^*(u^{n-1}) \in A^{n-1}. \tag{2}$$

By multiplying (1) and (2) and applying (ii-b) with $\bar{w}_{n-3} = u^{n-3}$, and (i-b), we obtain:

$$a \cup p^*(u^n) + 2ka \cup p^*(u^n) = 0. \tag{3}$$

Since $2k = 0$, we find that $a \cup p^*(u^n) = 0$. But, by (ii-a), this is a contradiction and (A) is proved.

PROOF OF B. We first prove a lemma.

LEMMA. *Every 2-plane bundle B over P^n decomposes into a Whitney sum of line bundles, if $n > 2$.*

PROOF. Suppose $w_2(B) = 0$.

Since $H^2(P^n) = Z_2$, if B is orientable, this is the first obstruction to a cross-section. If B is nonorientable, the obstruction is in $H^2(P^n)$ with *twisted* integer coefficients, which is zero. Since all the higher obstructions are zero, B has a cross-section and the lemma follows. If $w_2(B) \neq 0$, then, by the Wu formula:

$$Sq^1 w_2(B) = w_1(B)w_2(B) + w_3(B) \quad \text{(see [8])}$$

and $w_3(B) = 0$, we see that $w_1(B) = 0$. Now consider $B \otimes L$ where L is the nontrivial line bundle over P^n. By formula III of 4.4.3 of [3], which holds, by analogous considerations, for w_i:

$$w_2(B \otimes L) = w_2(B) + w_1(B)w_1(L) + w_1(L)^2.$$

Therefore $w_2(B \otimes L) = 0$ and $B \otimes L$ decomposes into line bundles. But then so does $B = (B \otimes L) \otimes L$.

Suppose P^n is immersed in E^{n+2}. By the lemma, the normal bundle is a sum of line bundles. Thus the *stable* normal bundle of P^n is a multiple of L, kL, where $k = 0$, 1 or 2. Since the stable tangent bundle of P^n is $(n+1)L$, $(n+k+1)L$ is the trivial bundle. But this contradicts the computations of Adams [1].

PROOF OF C. Let P^8 be immersed in E^{k+8} for k large, with normal k-plane bundle B. The group of stable vector bundles over P^n has been calculated by Adams [1]; for $n = 8$, it is the cyclic group of order 16 generated by L. It is well known that the stable tangent bundle of P^8 is $9L$; therefore the stable class of B is $7L$. This means B is the Whitney sum of seven copies of L and a trivial bundle, if k is large enough. Then it follows from 6.4 of [2] that there is an immersion of P^8 in E^{15} with normal bundle a Whitney sum of seven copies of L. Considering one of these copies of L as a tubular neighborhood of P^8 in P^9, we can immerse P^9-x in E^{15}. By 3.9 of [2], the obstruction to extending this immersion to P^9 in E^{15} is an element of $\pi_8(V_{15,9})$, which is zero by [7]. This completes the proof of (C).

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