A NOTE ON METRIZATION OF MOORE SPACES

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J. N. Younglove recently has proved [3] that a normal, complete Moore space is metrizable, provided that the boundary of each domain is screenable. It is established in the first theorem of this paper that the same conclusion follows without the requirement of completeness on the space. The second theorem establishes a relation between separability and pointwise paracompactness in complete Moore spaces.

The statement that the space S is a Moore space means there is a sequence of collections of regions in S satisfying Axiom 0 and Axiom 1_3 of [2]. A Moore space is complete if and only if there is a sequence of collections of regions in the space satisfying all of Axiom 1 of [2].

The statement that the point set M is screenable means that if G is an open covering of the space S, there exists a sequence H_1, H_2, $H_3, \cdots$ of collections of mutually exclusive open sets such that $\sum H_i$ is a refinement of G which covers M.

The statement that the point set M is pointwise paracompact means that if G is an open covering of the space S, there is an open refinement H of G such that H covers M and no point of S belongs to infinitely many elements of H.

THEOREM 1. *A normal Moore space is metrizable if and only if the boundary of each domain is screenable.*[1]

PROOF. Suppose S is a normal Moore space and H is a covering of S. Denote by H' a collection of mutually exclusive domains such that H'^* is dense in S and H' refines H. But H'^* is a domain so $S-H'^*$ is screenable. Then there exists a sequence $H_1, H_2, H_3, \cdots$ such that each H_i is a collection of mutually exclusive domains, each of which is a subset of some element of H, and $\sum H_i$ covers $S-H'^*$. In this case, $H', H_1, H_2, H_3, \cdots$ screens S with respect to H and thus, S is a screenable space. Bing has proved [1] that each normal, screenable space is metrizable.

Since Bing has proved also [1] that each metrizable Moore space is screenable, it follows that the boundary of each domain of a metrizable Moore space is screenable.

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[1] It is my understanding that E. E. Grace has an independent but similar proof of this theorem.

THEOREM 2. *A separable, complete Moore space is metrizable if and only if the boundary of each domain is pointwise paracompact.*

PROOF. Suppose S is a separable, complete Moore space. Younglove has proved [4] that there exists a metrizable subspace S' which is dense in S. The point set h is a domain in S' if and only if there exists a domain g in S such that h is $g \cdot S'$.

Suppose G is an open covering of S and H is the collection to which h belongs if and only if there is an element g of G such that h is $g \cdot S'$. In S', H is an open covering of S'. A theorem proved by Bing [1] establishes that there is a sequence H_1', H_2', H_3', $\cdots$ of discrete (in S') collections of open sets such that the sum of the collections of the sequence is a refinement of H which covers S'. For each element h' of each H_i', denote by h some domain in S such that $h \cdot S'$ is h'. If H_i is the collection to which the domain d belongs if and only if there is a domain h' of H_i' such that d is h, then H_i is a collection of mutually exclusive domains. For suppose each of h and h_1 belong to H_i and h intersects h_1. Then $h \cdot h_1$ is open and $h \cdot S'$ intersects $h_1 \cdot S'$. This means that H_i' is not a discrete collection of domains in S' and a contradiction is reached.

Since S is separable, each H_i is a countable collection. Since $S - \sum H_i^*$ is pointwise paracompact, there is a refinement G' of G covering $S - \sum H_i^*$ such that no point of S belongs to infinitely many of the elements of G'. But S is separable, so G' must be countable.

It is clear that if H' is the collection to which the domain d belongs if and only if d is a domain of G' or of some H_i, then H' is countable. Thus, S is completely separable, and therefore metrizable.

If the space is metrizable, it is paracompact. In this case, it is obvious that the boundary of each domain is pointwise paracompact.

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