THE CIRCULATION OF VECTOR FIELDS

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1. **Introduction.** Operating in Euclidean three-space, E^3, with u, v, w designating vectors and e, t, n designating unit vectors, let $v(p)$ be a continuous vector field defined in the neighborhood of the point p_0. Also let $C_n(p_0, r)$ represent the circle with center p_0 and radius r, lying in the plane through p_0 normal to n, oriented by the usual right-hand rule. Define the upper circulation per unit area of v at p_0 in the direction n, designated by $D_n^* v(p_0)$, as follows:

$$D_n^* v(p_0) = \limsup_{r \to 0} (\pi r^2)^{-1} \int_{C_n(p_0, r)} v \cdot t ds,$$

where t designates the unit tangent and ds the differential of arc length. Similarly define the lower circulation per unit area, $D_{*n} v(p_0)$, using lim inf. If $D_{*n} v(p_0) = D_n^* v(p_0)$ and both expressions are finite, designate this common value by $D_n v(p_0)$ and call it the circulation per unit area of v at p_0 in the direction n.

The curl of v is said to exist at the point p_0 if $D_n v(p_0)$ exists for every unit vector n and if, furthermore, there exists a vector w such that $w \cdot n = D_n v(p_0)$ for every unit vector n, w is then called curl $v(p_0)$.

The curl of v will be said to exist uniformly at p_0, if curl $v(p_0)$ exists and if, furthermore,

$$\lim_{r \to 0} (\pi r^2)^{-1} \int_{C_n(p_0, r)} v \cdot t ds = n \cdot \operatorname{curl} v(p_0),$$

uniformly in n.

It is clear that if $v(p)$ is in class C^1 in a neighborhood of the point p_0, then the curl of v exists uniformly at the point p_0.

The above definitions are classical and can be found in most of the standard books on advanced calculus. In particular, pictorial illustrations of the above discussion can be found in [1, p. 351; 2, p. 278].

The following theorem will be proved in this paper:

THEOREM. *Let $v(p)$ be a continuous vector field defined in an open set $R \subset E^3$. Suppose there exists three mutually orthogonal unit vectors e_1, e_2, e_3, and a constant K such that $|D_{e_j}^* v(p)| \leq K$ and $|D_{*e_j} v(p)| \leq K$ for p in R and $j = 1, 2, 3$. Then curl v exists uniformly almost everywhere in R.*

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The above theorem is the vector analogue of the classical theorem of Rademacher concerning Lipschitz functions and total differentials. (See [**3**, p. 310] or [**6**, p. 272].)

2. **Proof of theorem.** With no loss in generality, we can assume that E^3 has the usual Cartesian coordinate system with p in E^3 given by $p=(x_1, x_2, x_3)$ and that e_j is the unit vector e_j in the direction of the x_j-axis, $j=1, 2, 3$. Furthermore, we can assume that R is the interior of the unit ball in E^3. Then it follows from the proof in [**5**] (by reading σ as a disc instead of a two-simplex) that with K as in the hypothesis of the theorem that

$$\left|\int_{C_n(p,r)} v\cdot t ds\right| \leqq 3K\pi r^2 \qquad \text{if } C_n(p, r) \subset R. \tag{1}$$

The first lemma we prove is the following:

Lemma 1. *$D_n^*v(p)$ and $D_{*n}v(p)$ are bounded (by $3K$) Borel measurable functions in R which are equal almost everywhere in R.*

Let $A=\{p\mid D_n^*v(p)<a\}$, and let $B(p, r)$ be the closed three-ball with center p and radius r. Fix r_1 with $0<r_1<1$, and let k be an integer greater than $(1-r_1)^{-1}$. Then denote by A^k the following set:

$$A^k = \left\{p \mid (\pi r^2)^{-1}\int_{C_n(p,r)} v\cdot t ds \leqq a - k^{-1} \text{ for } 0 < r \leqq k^{-1} \text{ and } p \text{ in } B(0, r_1)\right\}\cdot$$

Since for fixed $r\leqq k^{-1}$, $(\pi r^2)^{-1}\int_{C_n(p,r)} v\cdot t ds$ is a continuous function of p in $B(0, r_1)$, we conclude that A^k is a closed set. But then $AB(0, r_1)$ is an F_σ-set, and consequently $D_n^*v(p)$ is a Borel measurable function in R. A similar proof shows that $D_{*n}v(p)$ is a Borel measurable function in R.

From (1), it follows that both $D_{*n}v(p)$ and $D_n^*v(p)$ are bounded by $3K$ in R. To show that $D_{*n}v(p)=D_n^*v(p)$ almost everywhere in R, set $u(p)=v(p)-[n\cdot v(p)]n$, and let $C_n(p_0, r_1)\subset R$. Then with $D_n(p_0, r_1)$, designating the closed disc having $C_n(p_0, r_1)$ as its boundary, we observe that $u(p)$ can be considered as a two-dimensional vector field defined on the two-dimensional set $D_n(p_0, r_1)$. Furthermore, it follows from (1) that for p in $D_n(p_0, r_1)$, $\limsup_{r\to 0} (\pi r^2)^{-1}\left|\int_{C_n(p,r)} u\cdot t ds\right| \leqq 3K$. Consequently, by [**4**, Theorem 2], for almost every p in $D_n(p_0, r_1)$,

$$\limsup_{r\to 0} (\pi r^2)^{-1}\int_{C_n(p,r)} u\cdot t ds = \liminf_{r\to 0} (\pi r^2)^{-1}\int_{C_n(p,r)} u\cdot t ds. \tag{2}$$

But $D_n^*v(p)$ and $D_{*n}v(p)$ are respectively the right and left sides of (2), and the conclusion of the lemma therefore follows from Fubini's theorem and the fact that $D_n^*v(p)$ and $D_{*n}v(p)$ are Borel measurable functions in R.

Next, for $f(p)$ a Borel measurable function defined almost everywhere in R which is, furthermore, essentially bounded on every compact subset of R, we define for p in $B(0, r)$ where $0<r<1$ and $0<h<1-r$, the function $f_h(p)=(4\pi h^3/3)^{-1}\int_{B(0,h)}f(p+q)dq$. We say that $f(p)$ is mean-continuous at p_0 if $f(p)$ is defined at p_0 and if $f_h(p_0)\to f(p_0)$ as $h\to 0$. It is clear that for fixed h, $f_h(p)$ is continuous in $B(0, r)$. Furthermore, it is well known that if $f(p)$ is continuous in R, then $f_h(p)$ is in class C^1 in $B(0, r)$.

We next define for p in $B(0, r)$, with h and r as above, the vector field $v_h(p)$ to be the vector field whose e_jth component at p is $[v\cdot e_j]_h(p)$, $j=1, 2, 3$. It is clear that $v_h(p)$ is in class C^1 in $B(0, r)$. Also, the following lemma prevails.

LEMMA 2. *For p_0 in $B(0, r)$, $D_nv_h(p_0)=[D_nv]_h(p_0)$.*

We first observe that for p in $B(0, r+r_1)$, where $r+r_1+h<1$, $v_h(p)\cdot t(p)=(4\pi h^3/3)^{-1}\int_{B(0,h)}v(p+q)\cdot t(p)dq$. Consequently,

$$\int_{C_n(p_0,r_1)} v_h(p)\cdot t(p)ds(p)$$

(3)
$$= (4\pi h^3/3)^{-1}\int_{B(0,h)}\left[\int_{C_n(p_0,r_1)} v(p+q)\cdot t(p)ds(p)\right]dq.$$

Dividing both sides of (3) by πr_1^2 and passing to the limit as $r_1\to 0$, we obtain $D_nv_h(p_0)$ from the left side of (3). From (1), Lemma 1, and the Lebesgue bounded convergence theorem, we obtain $[D_nv]_h(p_0)$ from the right side of (3), and Lemma 2 is established.

By Lemma 1, $D_nv(p)$ is a bounded Borel function defined almost everywhere in R. Set $Q_n=\{p\mid p$ in R and D_nv is mean-continuous at $p\}$. By Lebesgue's theorem, $R-Q_n$ is a set of Lebesgue measure zero. The following lemma then holds:

LEMMA 3. *For p_0 in $Q_{e_1}Q_{e_2}Q_{e_3}Q_n$,*

(4)
$$D_nv(p_0) = \sum_{j=1}^{3} n\cdot e_jD_{e_j}v(p_0).$$

Since for h small, $v_h(p)$ is in class C^1 in a neighborhood of p_0, $D_nv_h(p_0)=n\cdot\text{curl } v_h(p_0)=\sum_{j=1}^3 n\cdot e_jD_{e_j}v_h(p_0)$. But then by Lemma 2,

$$[D_n v]_h(p_0) = \sum_{j=1}^{3} n \cdot e_j [D_{e_j} v]_h(p_0). \tag{5}$$

Using the fact that p_0 is in $Q_{e_1}Q_{e_2}Q_{e_3}Q_n$, (4) follows immediately from (5) on passing to the limit as $h \to 0$.

By a spherical lune, σ, we shall mean one of the four possible two-dimensional closed sets determined by two great circles on a sphere. $|\sigma|$ will designate the two-dimensional area of σ, and $\partial\sigma$ will be oriented with respect to the outer normal of the sphere.

LEMMA 4. *Let $\sigma \subset \partial B(p, r)$ where $B(p, r) \subset R$. Then with K as in the hypothesis of the theorem, $|\int_{\partial\sigma} v \cdot t ds| \leqq 3K|\sigma|$.*

Lemma 4 follows in the same manner as the analogous result was established for simplices in [5, p. 85]. We need only observe that on setting $Q_1 = Q_{e_1}Q_{e_2}Q_{e_3}$, there exists a sequence of spherical lunes $\{\sigma_j\}_{j=1}^{\infty}$ contained in a small neighborhood of σ with the following properties: (a) $|\sigma_j| \to |\sigma|$, (b) $\int_{\partial\sigma_j} v \cdot t ds \to \int_{\partial\sigma} v \cdot t ds$ and (c) the two-dimensional measure of $Q_1\sigma_j$ on σ_j is the same as $|\sigma_j|$.

Before proving the theorem, we note that if n and n_0 are two unit vectors with $|n - n_0| < \epsilon$, $\epsilon > 0$ and small, and if σ_1 and σ_2 are the two smaller spherical lunes determined by $C_n(p, r)$ and $C_{n_0}(p, r)$, then

$$|\sigma_k| \leqq \epsilon \pi r^2 \qquad \text{for } k = 1, 2. \tag{6}$$

We now prove the theorem. Select a countable set $\{n_j\}_{j+1}^{\infty}$ of unit vectors with $n_j = e_j$ for $j = 1, 2, 3$, which is dense in the set of all unit vectors, and set $Q = \prod_{j=1}^{\infty} Q_{n_j}$. Clearly, $R - Q$ is of Lebesgue measure zero.

Next, let p be given in Q and define $w(p) = \sum_{j=1}^{3} D_{e_j} v(p) e_j$. We propose to show that given $\epsilon > 0$ and small, there exists r_0 (which depends on p and ϵ) such that if $0 < r \leqq r_0$, then for every unit vector n

$$\left| (\pi r^2)^{-1} \int_{C_n(p,r)} v \cdot t ds - n \cdot w(p) \right| < (9K + 1)\epsilon, \tag{7}$$

where K is the constant in the hypothesis of the theorem.

To do this, choose M so large that $\{n_j\}_{j=1}^{M}$ constitutes an ϵ-dense set. By Lemma 3 and the choice of Q, there exists an r_0 such that if $0 < r \leqq r_0$,

$$\left| (\pi r^2)^{-1} \int_{C_{n_j}(p,r)} v \cdot t ds - n_j \cdot w(p) \right| < \epsilon \qquad \text{for } j = 1, \cdots, M. \tag{8}$$

Next, given a unit vector n, choose n_j such that $|n - n_j| < \epsilon$. If

σ_1 and σ_2 are the spherical lunes in (6) with n_j playing the role of n_0, we then obtain from (6) and Lemma 4, that for $0 < r \leqq r_0$,

$$\left| \int_{C_n(p,r)} v \cdot t ds - \int_{C_{n_j}(p,r)} v \cdot t ds \right| \leqq \sum_{k=1}^{2} \left| \int_{\partial \sigma_k} v \cdot t ds \right| \leqq 6K\pi r^2 \epsilon.$$

But then from (8), we see that for $0 < r \leqq r_0$ the left side of (7) is majorized by $(6K+1)\epsilon + |(n - n_j) \cdot w(p)|$ which in turn is majorized by $(9K+1)\epsilon$. The inequality in (7) is consequently established, and the theorem is proved with curl $v(p) = w(p)$.

In closing, we remark that if S is a simple, oriented, C^1 surface contained in R and $S - SQ$ is of measure zero with respect to the natural measure on S, it is not difficult to see from [4] and [5], that Stokes' theorem holds for S with respect to v and curl v.

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