ON FINITELY ADDITIVE SET FUNCTIONS

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1. **Introduction.** Let $\mathfrak{F}$ be a σ-field of subsets of a set X, and let $\mathrm{ba}(X, \mathfrak{F})$ be the Banach space of all bounded real valued finitely additive set functions on $\mathfrak{F}$, with total variation norm. Let $\mathrm{ca}(X, \mathfrak{F})$ be the members of $\mathrm{ba}(X, \mathfrak{F})$ which are countably additive, and let $\mathrm{pf}(X, \mathfrak{F})$ be the members of $\mathrm{ba}(X, \mathfrak{F})$ which are purely finitely additive. Then $\mathrm{ba}(X, \mathfrak{F}) = \mathrm{ca}(X, \mathfrak{F}) \oplus \mathrm{pf}(X, \mathfrak{F})$ [**1**, p. 163; **2**].

Let X^* be the Boolean space of the Boolean algebra $\mathfrak{F}$. We assume without essential loss of generality that $\mathfrak{F}$ separates the points of X, so that X may be regarded as a dense subset of X^*. Space $\mathrm{ba}(X, \mathfrak{F})$ is isometrically isomorphic to the space $C^*(X^*)$ of bounded real valued signed Baire measures on X^*. Solving a problem raised in [**2**], we characterize $\mathrm{pf}(X, \mathfrak{F})$ as the set functions whose correspondents in $C^*(X^*)$ live on a Baire F_σ set of first category in X^*; such a set is necessarily contained in $X^* - X$, as it turns out. Further results are given for the case where each point of X is a measurable set.

2. **Preliminaries.** Let $B = B(X, \mathfrak{F})$ be the Banach algebra of bounded real valued $\mathfrak{F}$ measurable functions on X, with point-wise operations and supremum norm. We represent the conjugate space as $B^* = \mathrm{ba}(X, \mathfrak{F})$.

Let $X^* \subset B^*$ be the set of homomorphisms of B onto the reals, with the relative B^*/B (weak*) topology. Then B is isometrically algebraically isomorphic to the algebra $C(X^*)$ of all continuous real valued functions on compact Hausdorff X^*. If $\hat{f} \in C(X^*)$ corresponds to $f \in B$ then $\hat{f}$ is simply the unique continuous extension of f to all of X^*, regarding X as a (dense) subset of X^*. In particular, to each $E \in \mathfrak{F}$ there corresponds the open closure $\overline{E}$ of E in X^*, and each open and closed set in X^* is of the form $\overline{E}$ for some $E \in \mathfrak{F}$. The sets $\overline{E}$, $E \in \mathfrak{F}$, are a base for the topology of X^*. The correspondence $E \leftrightarrow \overline{E}$ preserves finite set operations, and $E = \overline{E} \cap X$, $E \in \mathfrak{F}$.

The smallest σ-field containing all $\overline{E}$, $E \in \mathfrak{F}$, is the σ-field $\mathfrak{B}$ of Baire subsets of X^*. We represent $C^*(X^*)$ as the space of bounded real valued signed Baire measures on X^*. If $\hat{\lambda} \in C^*(X^*)$ corresponds to $\lambda \in B^*$ then $\hat{\lambda}$ is determined by the values $\hat{\lambda}(\overline{E}) = \lambda(E)$, $E \in \mathfrak{F}$.

3. **Representation.**

THEOREM 1. *Let $F \subset X^*$ be a closed Baire set. Then F is nowhere dense if and only if $F \subset X^* - X$.*

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PROOF. Suppose F is a closed subset of X^*-X. If F contained a neighborhood then F would contain points of X, since X is dense. Thus F is nowhere dense. Conversely, let F be a closed nowhere dense Baire set. Then F is a G_δ: $F=\bigcap_1^\infty G_n$, G_n open [4, p. 221]. For each n there is a cover of the compact set G_n' by open and closed sets disjoint from F. Hence there is an open and closed set $\overline{E}_n$, $E_n\in\mathfrak{F}$, such that $F\subset\overline{E}_n\subset G_n$. Thus $F=\bigcap_1^\infty \overline{E}_n$, $F\cap X=\bigcap_1^\infty E_n$. If $E=\bigcap_1^\infty E_n\in\mathfrak{F}$ were nonempty then F would contain the neighborhood $\overline{E}$, and it follows that $F\subset X^*-X$. (In general, $\overline{E}$ is the open and closed interior of the closed Baire set F; cf. Theorem 4.11 of [2].)

The following result is Theorem 4.14 of [2].

THEOREM 2. *Suppose $\lambda\in B^*$. Then for $\lambda\in\mathrm{ca}(X, \mathfrak{F})$ to hold it is necessary and sufficient that $\hat{\lambda}(F)=0$ for each closed nowhere dense Baire set F.*

Theorems 1 and 2 show that in the sense of outer measure, X is thick in X^* relative to any $\hat{\lambda}$ such that $\lambda\in\mathrm{ca}(X, \mathfrak{F})$ [4, p. 74]. As we shall see, however, this does not imply that the members of $\mathrm{ca}(X, \mathfrak{F})$ have support X in X^*.

We denote by $\mathcal{C}$ the class of first category Baire F_σ sets, i.e., countable unions of closed nowhere dense Baire sets. For $\hat{\lambda}\in C^*(X^*)$ and $A\in\mathcal{B}$ we denote by $\hat{\lambda}_A$ the measure $\hat{\lambda}_A(E)=\hat{\lambda}(E\cap A)$, $E\in\mathcal{B}$.

THEOREM 3. *Suppose $\lambda\in B^*$. Then for $\lambda\in\mathrm{pf}(X, \mathfrak{F})$ to hold it is necessary and sufficient that $\hat{\lambda}=\hat{\lambda}_{C_0}$ for some $C_0\in\mathcal{C}$.*

PROOF. Sufficiency is a consequence of Theorem 4.16 of [2]. Conversely, suppose $\lambda\geqq 0$, $\lambda\in\mathrm{pf}(X, \mathfrak{F})$. Let number m be defined by

$$m = \sup_{C\in\mathcal{C}} \hat{\lambda}(C).$$

Let $\{C_n\}$ be a sequence in $\mathcal{C}$ such that $\lim \hat{\lambda}(C_n)=m$. Then $C_0=\bigcup_1^\infty C_n$ is a member of $\mathcal{C}$ and $\hat{\lambda}(C_0)=m$. Consider the measure $\hat{\mu}=\hat{\lambda}-\hat{\lambda}_{C_0}$. We have $\hat{\mu}(C)=0$, $C\in\mathcal{C}$, clearly, so that $\mu\in\mathrm{ca}(X, \mathfrak{F})$, from Theorem 2. Since $0\leqq\mu\leqq\lambda$ and $\lambda\in\mathrm{pf}(X, \mathfrak{F})$, we must have $\mu=0$, and the theorem follows.

We assume from now on that each point of X is measurable; $\{x\}\in\mathfrak{F}$ for each $x\in X$. Then each point $x\in X$ is an open and closed set in X^*, X is open and $Y=X^*-X$ is closed in X^*. Suppose $\lambda=\lambda_1+\lambda_2$, $\lambda_1\in\mathrm{ca}(X, \mathfrak{F})$, $\lambda_2\in\mathrm{pf}(X, \mathfrak{F})$, λ_1, $\lambda_2\geqq 0$. Denote by $\tilde{\lambda}$ the regular Borel extension of $\hat{\lambda}$. Then with $\tilde{\lambda}_3(E)=\tilde{\lambda}(E\cap X)$, $\tilde{\lambda}_4(E)=\tilde{\lambda}(E\cap Y)$, Borel $E\subset X^*$, we have $\tilde{\lambda}=\tilde{\lambda}_3+\tilde{\lambda}_4$. It is clear from Theorems 1 and 2 that $\lambda_3 \in \mathrm{ca}(X, \mathfrak{F})$; let us see what this measure is. Since $\hat{\lambda}_3$ is regular, we have

$$\|\hat{\lambda}_3\| = \hat{\lambda}_3(X) = \sup_{C \in X} \hat{\lambda}_3(C),$$

where the supremum is over all compact subsets of X in X^*. But the only compact subsets of X in X^* are the finite subsets of X. As in the proof of Theorem 3, we find a countable subset A_0 of X such that $\hat{\lambda}_3(E) = \hat{\lambda}_3(E \cap A_0)$, Borel $E \subset X^*$. Thus λ_3 is the purely atomic part of λ_1. The atomless part of λ_1 is contained in λ_4, living on Y; this is $\lambda_5 = \lambda_4 - \lambda_2$. Let $\hat{\lambda}_*$, $\hat{\lambda}^*$ denote the inner and outer measures induced by $\hat{\lambda}$. Then $\hat{\lambda}_*(Y) = \|\lambda_2\|$, $\hat{\lambda}^*(Y) = \|\lambda_2\| + \|\lambda_5\|$, whence the Borel set Y is not $\hat{\lambda}^*$ measurable when $\lambda_5 \neq 0$. A fortiori, $\hat{\lambda}$ is not completion regular when $\lambda_5 \neq 0$ [**4**, p. 230].

To summarize, $\lambda \in B^*$ has the decomposition $\lambda = \lambda_2 + \lambda_3 + \lambda_5$, where $\lambda_2 \in \mathrm{pf}(X, \mathfrak{F})$ corresponds to $\hat{\lambda}_2$ living on a Baire F_σ subset of $X^* - X$, where the purely atomic part λ_3 corresponds to $\hat{\lambda}_3$ living on a countable subset of X in X^*, and where the atomless countably additive part λ_5 corresponds to $\hat{\lambda}_5$ which lives on $X^* - X$ and assigns measure 0 to every Baire F_σ subset of $X^* - X$. (It is easily seen that $X^* - X$ is a Baire set if and only if X is countable. There are no atomless countably additive measures in this case, of course.)

4. **Measurable cardinals.** The above results throw some light on, but do not solve, the problem of the existence of measurable cardinals [**3**, Chapter 12]. Let X be a discrete space of cardinal $|X|$, and let $\mathfrak{F}$ be the field of all subsets of X. Then $X^* = \beta X$ is the Stone-Čech compactification of X. For each $x \in X^*$ denote by $\lambda_x \in \mathrm{ba}(X, \mathfrak{F})$ the set function corresponding to the unit point measure at x: $\hat{\lambda}_x(E) = \chi_E(x)$, $E \in \mathfrak{B}$, where χ_E is the characteristic function of E. The cardinal $|X|$ is said to be measurable if $\lambda_x \in \mathrm{ca}(X, \mathfrak{F})$ for some $x \in X^* - X$.

Denote by $\Gamma = \bigcup_{C \in \mathfrak{C}} C$ the union of all closed Baire subsets of $X^* - X$, and define $\Delta = (X^* - X) - \Gamma$.

THEOREM 4. *The cardinal $|X|$ is measurable if and only if Δ is nonempty.*

PROOF. By Theorems 1 and 2, $\lambda_x \in \mathrm{ca}(X, \mathfrak{F})$ if and only if x is contained in no closed Baire subset of $X^* - X$.

A point $x \in \Delta$ has the property that x is an interior point of every closed Baire set containing it. Dually, if $x \in \Delta$ is disjoint from an open Baire set U then x is disjoint from the open closure $\overline{U}$ of U. It is easy to show that every neighborhood of $x \in \Delta$ meets X in a set of measurable cardinal.

Added in proof. Theorem 4 is not new. If X is any completely regular space, then $\beta X - \upsilon X$ is the union of all Baire subsets of βX disjoint from X; cf. [**3**, 119].

References

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