ON A THEOREM OF P. J. COHEN AND H. DAVENPORT

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Let G be a compact Abelian group with character group X. P. J. Cohen [1] has proved that if X is torsion-free, if Υ is a finite subset of X consisting of N characters, and $|\alpha_\chi| \geqq 1$ for all $\chi \in \Upsilon$, then

$$(0) \qquad \int_G \left| \sum_{\chi \in \Upsilon} \alpha_\chi \chi(x) \right| dx > K \left(\frac{\log N}{\log \log N} \right)^{1/8},$$

where the integral is the Haar integral on G, K is some positive constant not depending on G, and N is sufficiently large. For the case in which G is the circle group, H. Davenport [2] has improved (0) by replacing the exponent $\frac{1}{8}$ by $\frac{1}{4}$ and the constant K by $\frac{1}{8}$. Cohen's and Davenport's arguments can in all likelihood be combined to yield (0) with exponent $\frac{1}{4}$ for an arbitrary G such that X is torsion-free.

In this note we apply Davenport's ideas to prove (0) with exponent $\frac{1}{4}$ not only for the case of torsion-free X but also for the case in which the torsion subgroup of X is an arbitrary finite Abelian group. By using care in our estimates we find some fairly large possible K's, and we also work out some numerical cases. If X has infinite torsion subgroup, we show that no inequality like (0) can possibly hold.

THEOREM A. *Let G be a compact Abelian group with character group* X. *Suppose that the torsion subgroup of* X *is finite and consists of f elements. Let Υ be a set of N distinct elements of* X, *where N is a positive integer. For each $\chi \in \Upsilon$, let α_χ be a complex number such that $|\alpha_\chi| \geqq 1$. Then for every number $K < (1-e^{-2})6^{-1/2}$, we have*

$$(1) \qquad \int_G \left| \sum_{\chi \in \Upsilon} \alpha_\chi \chi(x) \right| dx > K \left(\frac{\log N}{\log \log N} \right)^{1/4}$$

provided that N is sufficiently large, depending upon K and f. For example, if $K = 3/10$, (1) *holds for all N such that either*

$$(2) \qquad \begin{aligned} &N > e^{310} \quad \text{and} \quad f \leqq 6 \text{ or} \\ &N \geqq \left(\frac{3}{2} f^2 \right)^{3f^2} \quad \text{and} \quad f > 6. \end{aligned}$$

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PROOF. Throughout the proof, we suppose that

$$N > f. \tag{3}$$

Let Φ be the subgroup of X generated by Υ. Then Φ is isomorphic with a direct product of finitely many cyclic groups, say a infinite cyclic groups and b finite cyclic groups. Thus every $\chi \in \Upsilon$ corresponds to a unique sequence

$$(c_1, c_2, \cdots, c_a, c_{a+1}, \cdots, c_{a+b}), \tag{4}$$

where $c_1, c_2, \cdots, c_a$ are arbitrary integers and $c_{a+1}, c_{a+2}, \cdots, c_{a+b}$ are nonnegative integers less than some fixed positive integers. We have $b \geqq 0$ and $a > 0$ since $N > f$.

We impose a complete ordering on Φ. If χ and χ' are distinct elements of Φ, if χ corresponds to $(c_1, c_2, \cdots, c_{a+b})$, and χ' corresponds to $(c_1', c_2', \cdots, c_{a+b}')$, then we write $\chi < \chi'$ if $c_j < c_j'$ where j is the first index k for which c_k differs from c_k'. We now write Υ as

$$\{\chi_1, \chi_2, \cdots, \chi_N\} \qquad \text{where } \chi_1 < \chi_2 < \cdots < \chi_N. \tag{5}$$

For $\chi \in \Phi$, let $N(\chi)$ be the cardinal number of the set $\{\psi \in \Upsilon : \psi \leqq \chi\}$.

Following Cohen's construction, we now define subsets $P_0, P_1, \cdots, P_k$ of Φ and subsets $T_0, T_1, \cdots, T_k$ of Υ, where k is a positive integer to be chosen later. Each set T_j is to consist of exactly r characters, where r is an integer >1. First, we set $P_0 = \{\chi_1\}$ and $T_0 = \varnothing$. Suppose that the sets $P_0, P_1, \cdots, P_{l-1}$ and the sets $T_0, T_1, \cdots, T_{l-1}$ have been defined. We determine the elements $\chi_{m_1^{(l)}}, \chi_{m_2^{(l)}}, \cdots, \chi_{m_r^{(l)}}$ of T_l as follows. We take $m_1^{(l)} = 1$. Suppose that the indices $m_1^{(l)}, m_2^{(l)}, \cdots, m_{j-1}^{(l)}$ have been determined ($j \leqq r$). Then we want $m_j^{(l)}$ to be the smallest index $m > m_{j-1}^{(l)}$ such that

$$\chi \chi_{m_i^{(l)}} \bar{\chi}_m \notin \Upsilon \tag{6}$$

for all $\chi \in P_{l-1}$ and all $i = 1, 2, \cdots, j-1$. Let us find conditions under which m exists.

Let χ, χ', and χ'' be any characters in Φ, and let $(c_1, c_2, \cdots, c_{a+b})$, $(c_1', c_2', \cdots, c_{a+b}')$, $(c_1'', c_2'', \cdots, c_{a+b}'')$ be the corresponding sequences (4). Now suppose that χ' and χ'' are such that $\chi' < \chi''$ and that q is the smallest index such that c_q' differs from c_q''. If q is less than or equal to a, then the inequality

$$\chi \chi' \bar{\chi}'' < \chi \tag{7}$$

obtains. We restrict m to be greater than or equal to $m_{j-1}^{(l)} + f$. Thus the inequality (7) applies to the product appearing in (6). Among these m's, we count those which must be rejected for violating (6).

For each i, we have to reject χ_m if

$$\chi_m = \chi\chi_{m_i^{(l)}}\bar{\psi}$$

where $\bar{\psi}\in\Upsilon$ and $\chi\in P_{l-1}$. Such a ψ must be less than χ in the ordering (5), and so we reject at most

$$\sum_{\chi\in P_{l-1}} N(\chi)$$

characters for each i. Consequently, taking $i=1, 2, \cdots, j-1$, we reject at most

$$(j-1)\sum_{\chi\in P_{l-1}} N(\chi)$$

characters. It follows that $m_j^{(l)}$ exists if

$$m_{j-1}^{(l)} + (f-1) + (j-1)\sum_{\chi\in P_{l-1}} N(\chi) \leqq N, \tag{8}$$

and that

$$m_j^{(l)} \leqq m_{j-1}^{(l)} + (f-1) + (j-1)\sum_{\chi\in P_{l-1}} N(\chi) \tag{9}$$

if (8) holds. Supposing that (8) holds for $j=r$ (and hence for $j=2, 3, \cdots, r-1$), we sum (9) over $j=2, 3, \cdots, i$ $(2\leqq i\leqq r)$ to obtain

$$m_i^{(l)} \leqq 1 + (i-1)(f-1) + \frac{i(i-1)}{1}\sum_{\chi\in P_{l-1}} N(\chi). \tag{10}$$

The inequality (10) also holds for $i=1$.

We next define the set $P_l\subset\Phi$:

$$P_l = P_{l-1} \cup \left\{\chi\chi_{m_i^{(l)}}\bar{\chi}_{m_j^{(l)}} : \chi \in P_{l-1};\ 1 \leqq i < j \leqq r\right\} \cup T_l. \tag{11}$$

The right side of (10) is obviously monotonic increasing in i and is also monotonic nondecreasing in l because $P_{l-1}\subset P_l$. Hence we can find the sets of characters $T_1, T_2, \cdots, T_k$ provided that

$$1 + (r-1)(f-1) + \frac{r(r-1)}{2}\sum_{\chi\in P_{k-1}} N(\chi) \leqq N. \tag{12}$$

Let us now use (11) to estimate the size of $\sum_{\chi\in P_{k-1}} N(\chi)$. For $l=1, 2, \cdots, k-1$, (11) implies that

$$\sum_{\chi\in P_l} N(\chi) \leqq \sum_{\chi\in P_{l-1}} N(\chi) + {\sum}' N(\chi\chi_{m_i^{(l)}}\bar{\chi}_{m_j^{(l)}}) + \sum_{\chi\in T_l} N(\chi), \tag{13}$$

where $\sum'$ denotes the sum over $\chi\in P_{l-1}$ and $1\leqq i<j\leqq r$. We chose

$m_1^{(l)}, m_2^{(l)}, \cdots, m_r^{(l)}$ in such a way that the relation (7) holds for the product $\chi\chi_{m_i^{(l)}}\bar{\chi}_{m_j^{(l)}}$. Hence we have

$$\sum{}' N(\chi\chi_{m_i^{(l)}}\bar{\chi}_{m_i^{(l)}}) \leqq \sum{}' N(\chi) \leqq \frac{r(r-1)}{2} \sum_{\chi \in P_{l-1}} N(\chi). \tag{14}$$

For $\chi_{m_i^{(l)}} \in T_l$, it is obvious that $N(\chi_{m_i^{(l)}}) = m_{i^{(l)}}$. Using this and (10), we write

$$\begin{aligned} \sum_{\chi \in T_l} N(\chi) &= \sum_{i=1}^{r} m_i^{(l)} \\ &\leqq \sum_{i=1}^{r} \left\{ 1 + (i-1)(f-1) + \frac{i(i-1)}{2} \sum_{\chi \in P_{l-1}} N(\chi) \right\} \\ &= r + \frac{r(r-1)}{2}(f-1) + \frac{r(r^2-1)}{6} \sum_{\chi \in P_{l-1}} N(\chi). \end{aligned} \tag{15}$$

Combining (13), (14), and (15), we see that

$$\begin{aligned} \sum_{\chi \in P_l} N(\chi) \leqq r &+ \frac{r(r-1)}{2}(f-1) \\ &+ \frac{r^3 + 3r^2 - 4r + 6}{6} \sum_{\chi \in P_{l-1}} N(\chi). \end{aligned} \tag{16}$$

The recurrence inequality (16) has the form

$$A_l \leqq a + bA_{l-1}, \qquad b > 1, \quad a > 0.$$

This implies that

$$\begin{aligned} A_l &\leqq b^l \left(A_0 + \frac{a}{b-1} \right) - \frac{a}{b-1} \\ &\leqq b^l \left(1 + \frac{a}{b-1} \right). \end{aligned} \tag{17}$$

From now on, suppose that

$$r \geqq \max(6, f) \quad \text{and} \quad k \text{ is an integer } \geqq 3. \tag{18}$$

The inequality (17) for $l = k-1$ is

$$\sum_{\chi \in P_{k-1}} N(\chi) = A_{k-1} \leqq b^{k-1}\left(1 + \frac{a}{b-1}\right),$$

where

$$b = \frac{r^3 + 3r^2 - 4r + 6}{6} \leqq \frac{1}{4} r^3,$$

$$b - 1 \geqq \frac{r^3}{6},$$

$$a = r + \frac{r(r-1)}{2}(f-1) \leqq fr^2.$$

Thus we have

$$\sum_{\chi \in \mathrm{P}_{k-1}} N(\chi) \leqq \left(\frac{r^3}{4}\right)^{k-1}\left(1 + \frac{6}{r^3} fr^2\right)$$

$$= \left(\frac{r^3}{4}\right)^{k-1}\left(1 + \frac{6f}{r}\right) \leqq r^{3(k-1)} \frac{7}{4^{k-1}} < r^{3(k-1)}.$$

We now return to relation (12). A routine computation shows that (12) holds if

$$r^{3k-1} \leqq N. \tag{19}$$

Thus we can find the sets of characters $\mathrm{T}_1, \mathrm{T}_2, \cdots, \mathrm{T}_k$ provided that (18) and (19) hold.

In determining k and r for which (19) holds, it is convenient to take $k = r^2$; and this choice of k turns out to be satisfactory for the final arguments of the present proof. Then we take

$$r = \left[\left(\frac{2 \log N}{3 \log \log N}\right)^{1/2}\right], \tag{20}$$

where $[\cdots]$ is the greatest integer symbol and

$$N \geqq e^e. \tag{21}$$

It is then easy to see that (19) holds.

We next construct by induction a sequence of functions $\phi_0, \phi_1, \cdots, \phi_k$ on the group G. Each χ in Υ is a certain χ_j under the ordering of (5); we write $\beta_j = \alpha_{\chi_j}|\alpha_{\chi_j}|^{-1}$. We define ϕ_0 to be the function $\beta_1\chi_1$. For $l \in \{1, 2, \cdots, k\}$, suppose that ϕ_{l-1} has been defined. Following Davenport [2], we define ϕ_l by the equality

$$\begin{aligned}\phi_l = \phi_{l-1}&\left\{1 - \frac{2}{r^2} - \frac{1}{r^3} \sum{}' \beta_{m_i^{(l)}}\chi_{m_i^{(l)}}\bar{\beta}_{m_j^{(l)}}\bar{\chi}_{m_j^{(l)}}\right\} \\ &+ \frac{1}{r^{5/2}} \sum_{j=1}^{r} \beta_{m_j^{(l)}}\chi_{m_j^{(l)}};\end{aligned} \tag{22}$$

in which the sum $\sum'$ is over all i, j such that $1 \leqq i < j \leqq r$.

Applying the method of Davenport's Lemma 2 [2], we make the following computations. First, for $x \in G$, write

$$\sum{}' \beta_{m_i^{(l)}} \bar{\beta}_{m_j^{(l)}} \chi_{m_i^{(l)}}(x) \bar{\chi}_{m_j^{(l)}}(x) = P + iQ.$$

Then it is clear that

$$P^2 + Q^2 \leqq \left(\frac{1}{2} r(r-1)\right)^2 < \frac{1}{4} r^4$$

and

$$\left| \sum_{j=1}^{r} \beta_{m_j^{(l)}} \chi_{m_j^{(l)}}(x) \right|^2 = r + 2 \operatorname{Re} \sum{}' \beta_{m_i^{(l)}} \bar{\beta}_{m_j^{(l)}} \chi_{m_i^{(l)}}(x) \bar{\chi}_{m_j^{(l)}}(x)$$
$$= r + 2P.$$

Hence we have

$$P \geqq \frac{r}{2} \cdot$$

Since $r \geqq 6$, Davenport's Lemma 1 shows that if $|\phi_{l-1}(x)| \leqq 1$, then

$$|\phi_l(x)| \leqq \left| 1 - \frac{2}{r^2} - \frac{P + iQ}{r^3} \right| + \frac{1}{r^{5/2}} (r + 2P)^{1/2} \leqq 1.$$

Since $|\phi_0| = |\beta_1 \chi_1| = 1$, it follows that $|\phi_l| \leqq 1$ $(l = 0, 1, \cdots, k)$.

We also follow Davenport to define

$$I_l = \int_G \phi_l(x) \sum_{\chi \in \mathrm{T}} \bar{\alpha}_\chi \bar{\chi}(x) dx. \tag{23}$$

The construction of $\mathrm{T}_1, \mathrm{T}_2, \cdots, \mathrm{T}_k$ shows that

$$I_l = \left(1 - \frac{2}{r^2}\right) I_{l-1} + \frac{1}{r^{5/2}} \sum_{j=1}^{r} \beta_{m_j^{(l)}} \bar{\alpha}_{\chi_{m_j^{(l)}}}$$
$$= \left(1 - \frac{2}{r^2}\right) I_{l-1} + \frac{1}{r^{5/2}} \sum{}' |\alpha_\chi|,$$

where $\sum'$ denotes the sum over $\chi = \chi_{m_j^{(l)}}$, $j = 1, 2, \cdots, r$. Since $|\alpha_\chi| \geqq 1$, we now have

$$I_l \geqq \left(1 - \frac{2}{r^2}\right) I_{l-1} + \frac{1}{r^{3/2}} \qquad (l = 1, 2, \cdots, k-1). \tag{24}$$

Thus we have

$$I_l - \frac{1}{2} r^{1/2} \geqq \left(1 - \frac{2}{r^2}\right)\left(I_{l-1} - \frac{1}{2} r^{1/2}\right),$$

so that

$$I_k - \frac{1}{2} r^{1/2} \geqq \left(1 - \frac{2}{r^2}\right)^k \left(I_0 - \frac{1}{2} r^{1/2}\right),$$

$$I_k \geqq \frac{1}{2} r^{1/2} - \left(1 - \frac{2}{r^2}\right)^k \left(\frac{1}{2} r^{1/2} - \left| \alpha_{\chi_1} \right| \right),$$

and

$$I_k \geqq \frac{1}{2} r^{1/2} - \left(1 - \frac{2}{r^2}\right)^k \left(\frac{1}{2} r^{1/2} - 1\right). \tag{25}$$

If $t > 2^{1/2}$, then

$$\left(1 - \frac{2}{t^2}\right)^{t^2} < \frac{1}{e^2} \cdot$$

hence for $k = r^2$, we have

$$I_k > \frac{1}{2} (1 - e^{-2}) r^{1/2}. \tag{26}$$

Since $|\phi_k| \leqq 1$, we have

$$\int_G \left| \sum_{\chi \in \Upsilon} \alpha_\chi \chi(x) \right| dx \geqq \left| \int_G \phi_k(x) \sum_{\chi \in \Upsilon} \bar{\alpha}_\chi \bar{\chi}(x) dx \right| \tag{27}$$

$$= I_k > \frac{1}{2} (1 - e^{-2}) r^{1/2}.$$

Using the value for r in (20), we obtain

$$\frac{1}{2} (1 - e^{-2}) r^{1/2} > \frac{1}{2} (1 - e^{-2}) \left(\left(\frac{2 \log N}{3 \log \log N} \right)^{1/2} - 1 \right)^{1/2} \tag{28}$$

$$= (1 - e^{-2}) 6^{-1/2} \left(1 - \left(\frac{3 \log \log N}{2 \log N} \right)^{1/2} \right)^{1/2} \left(\frac{\log N}{\log \log N} \right)^{1/4}.$$

Now let K be any number such that $K < (1 - e^{-2}) 6^{-1/2}$. Combining (27) and (28), we see that

$$\int_G \left| \sum_{\chi \in \Upsilon} \alpha_\chi \chi(x) \right| dx > K \left(\frac{\log N}{\log \log N} \right)^{1/4} \tag{29}$$

for N so large that conditions (3), (18), (21) hold and

$$(1 - e^{-2})6^{-1/2}\left(1 - \left(\frac{3 \log \log N}{2 \log N}\right)^{1/2}\right)^{1/2} \geqq K. \tag{30}$$

Note that r is defined by (20) and that $k=r^2$.

The condition $k\geqq 3$ is implied by $r\geqq 6$. The conditions (3) and (21) will be trivially satisfied by the following. Since 6 and f are integers, (18) will be satisfied if

$$\frac{\log N}{\log \log N} \geqq \frac{3}{2} \max (36, f^2) = \max\left(54, \frac{3}{2} f^2\right). \tag{31}$$

If K is taken less than $((1-e^{-2})/6)5^{1/2}$, say $K=3/10$, then (30) is implied by (31). To see how large N must be for (31) to hold, define u by

$$N = e^{2u \log u}.$$

Then

$$\frac{\log N}{\log \log N} = 2u\left(\frac{\log 2}{\log u} + 1 + \frac{\log \log u}{\log u}\right)^{-1} \geqq u,$$

since $\log \log u \leqq \log u - 1$. Hence if $N\geqq (3f^2/2)^{3f^2}$, we have

$$\frac{\log N}{\log \log N} \geqq \frac{3}{2} f^2.$$

Furthermore, if $N>e^{310}$, then $\log N/\log \log N>54$. Thus if $f\leqq 6$, (29) holds with $K=3/10$ for all $N>e^{310}$. The inequality (29) with $K=3/10$ also holds for $N\geqq (3f^2/2)^{3f^2}$ if $f>6$. This completes the proof of Theorem A.

THEOREM B. *Let G be a compact Abelian group with character group* X, *and let* Γ *be a finite subgroup of* X. *Then*

$$\int_G \left| \sum_{\chi\in\Gamma} \chi(x) \right| dx = 1. \tag{32}$$

PROOF. Let $A=\{x\in G: \chi(x)=1 \text{ for all } \chi\in\Gamma\}$. It is well known that the quotient group G/A is isomorphic with the character group of Γ. Let λ denote normalized Haar measure on G. Then we have $\lambda(A)=1/o(\Gamma)$. For $\chi_0\in\Gamma$ and $x\in G$, we have

$$\sum_{\chi\in\Gamma} \chi(x) = \sum_{\chi\in\Gamma} \chi_0\chi(x) = \chi_0(x) \sum_{\chi\in\Gamma} \chi(x).$$

If $\sum_{\chi\in\Gamma} \chi(x) \neq 0$, it follows that $\chi_0(x)=1$. Since χ_0 can be any element of Γ, we have $\sum_{x\in\Gamma}\chi(x)=0$ if $x\notin A$. It is trivial that $\sum_{x\in\Gamma}\chi(x)=o(\Gamma)$ if $x\in A$. The equality (32) follows at once.

Theorem B shows that the number N appearing in Theorem A must go to infinity as f goes to infinity. It also shows that Theorem A fails completely if the torsion subgroup of X is infinite. For in this case X contains finite subgroups of arbitrarily large order, and (32) shows that nothing like (1) can hold.

It should also be noted that if the torsion subgroup of **X** is finite, then it is a direct factor of **X**. First, the torsion subgroup is always a pure subgroup of X; and then one can quote, for example, a well-known theorem of Łoś [**3**, (25.21)]. Thus G is topologically the union of a finite number of replicas of a connected compact Abelian group. There appears to be no advantage in using this fact for our proof.

Bibliography

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