

WEAK CONVERGENCE OF BOUNDED SEQUENCES

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The purpose of this note is to give a criterion for the weak convergence of a bounded sequence in a Banach space. The author wishes to acknowledge extremely useful conversations with E. Bishop, W. Eberlein, K. de Leeuw, V. Pták and W. Rudin.

If B is a Banach space, let $U^* = \{f: f \in B^* \text{ and } \|f\| \leq 1\}$ be the unit ball of B^* and let E be the set of extreme points of U^* .

THEOREM. *Suppose that $\{x_n\}$ is a bounded sequence in the Banach space B , and let $x \in B$. Then $\{x_n\}$ converges weakly to x if (and only if) $\lim_{n \rightarrow \infty} \int f(x_n) = \int f(x)$ for each f in E .*

PROOF. Assume that $\|x_n\| \leq M$, $n = 1, 2, \dots$. It suffices to show that for each g in U^* , $\lim g(x_n) = g(x)$. Now, U^* is convex and compact (we only consider U^* in its weak* topology) and the function Q defined by $(Qx)(h) = h(x)$ for x in B , h in U^* , is a linear isometry of B into $C(U^*)$, the space of continuous functions on U^* with the supremum norm. Suppose that g is an element of U^* . By the Bishop-de Leeuw generalization [1, Theorem 5.6] of a well-known result of Choquet, there exists a σ -ring of subsets of U^* (generated by E and the Baire sets) and a non-negative finite measure μ on this σ -ring such that $\mu(U^*) = \mu(E)$ and $g(y) = \int_{U^*} Qy d\mu$ for each y in B . Since $\mu(U^* \setminus E) = 0$, it follows that the functions Qx_n converge to Qx almost everywhere with respect to μ . Since $|Qx_n| \leq \|Qx_n\| = \|x_n\| \leq M$, we conclude from Lebesgue's bounded convergence theorem that $g(x) = \int Qx d\mu = \lim \int Qx_n d\mu = \lim g(x_n)$, which completes the proof.

BIBLIOGRAPHY

1. E. Bishop and K. de Leeuw, *The representation of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier (Grenoble) 9 (1959), 305-331.

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