WEAK CONVERGENCE OF BOUNDED SEQUENCES

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The purpose of this note is to give a criterion for the weak convergence of a bounded sequence in a Banach space. The author wishes to acknowledge extremely useful conversations with E. Bishop, W. Eberlein, K. de Leeuw, V. Pták and W. Rudin.

If B is a Banach space, let $U^* = \{f: f \in B^* \text{ and } \|f\| \leqq 1\}$ be the unit ball of B^* and let E be the set of extreme points of U^*.

THEOREM. *Suppose that* $\{x_n\}$ *is a bounded sequence in the Banach space* B, *and let* $x \in B$. *Then* $\{x_n\}$ *converges weakly to* x *if* (*and only if*) $\lim_{n\to\infty} f(x_n) = f(x)$ *for each* f *in* E.

PROOF. Assume that $\|x_n\| \leqq M$, $n = 1, 2, \cdots$. It suffices to show that for each g in U^*, $\lim g(x_n) = g(x)$. Now, U^* is convex and compact (we only consider U^* in its weak* topology) and the function Q defined by $(Qx)(h) = h(x)$ for x in B, h in U^*, is a linear isometry of B into $C(U^*)$, the space of continuous functions on U^* with the supremum norm. Suppose that g is an element of U^*. By the Bishop-de Leeuw generalization [1, Theorem 5.6] of a well-known result of Choquet, there exists a σ-ring of subsets of U^* (generated by E and the Baire sets) and a non-negative finite measure μ on this σ-ring such that $\mu(U^*) = \mu(E)$ and $g(y) = \int_{U^*} Qy\,d\mu$ for each y in B. Since $\mu(U^* \sim E) = 0$, it follows that the functions Qx_n converge to Qx almost everywhere with respect to μ. Since $|Qx_n| \leqq \|Qx_n\| = \|x_n\| \leqq M$, we conclude from Lebesgue's bounded convergence theorem that $g(x) = \int Qx\,d\mu = \lim \int Qx_n\,d\mu = \lim g(x_n)$, which completes the proof.

BIBLIOGRAPHY

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