A NOTE ON BOUNDED-TRUTH-TABLE REDUCIBILITY

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1. **Introduction.** In the 1944 paper of Post [1], the notions of *one-one*, *many-one*, *bounded-truth-table*, *truth-table* and *Turing* reducibility are introduced. For sets A, B of positive integers let us abbreviate the statement "*A is one-one* (*many-one*, *bounded-truth-table*) *reducible to B*" by "$A \leqq_1 B$" ("$A \leqq_m B$," "$A \leqq_{btt} B$").

Bounded-truth-table and truth-table reducibilities are shown by Post to be distinct relations over the recursively enumerable, nonrecursive sets. In [2] and in [6], respectively, Dekker shows that one-one and many-one reducibilities differ on these sets and that truth-table and Turing reducibilities are distinct. This note will show that many-one and bounded-truth-table reducibilities also differ on these sets. Since the five reducibilities given are linearly ordered under implication (if $A \leqq_1 B$, then $A \leqq_m B$, if $A \leqq_m B$, then $A \leqq_{btt} B$, etc.), the conclusion that all five reducibilities are distinct on the recursively enumerable, nonrecursive sets will follow.

A second theorem will provide an example of a recursively enumerable bounded-truth-table degree of unsolvability which contains infinitely many distinct many-one degrees.

2. **Preliminaries.** Familiarity with §§1–8 of [1] will be assumed and the notation therein will be used. Let N denote the set of all positive integers. Let A^n denote the Cartesian product of a set A itself n times. Thus, A^n is the set of all ordered n-tuples $\langle x_1, x_2, \cdots, x_n \rangle$ of positive integers, all of whose components $\{x_i\}$ are in A.[2]

REMARK. It is clear from the definition of bounded-truth-table reducibility [1, p. 301] that for any set A and any $n \in N$, $A^n \leqq_{btt} A$ since $\langle x_1, x_2, \cdots, x_n \rangle \in A^n \equiv x_1 \in A \mathbin{\&} x_2 \in A \mathbin{\&} \cdots \mathbin{\&} x_n \in A$. Also, $A \leqq_1 A^n$; thus, the bounded-truth-table degree of unsolvability containing a set A must contain A^n for all $n \in N$.

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[2] When considering certain recursively invariant properties of A^n, one usually works with indices of the n-tuples of A^n under a fixed effective one-one mapping from N^n into N. However, we shall let the notation A^n stand for either the set of n-tuples of $A \times A \times \cdots \times A$ (n times) or the set of indices of those n-tuples, as context will make it clear which meaning is intended.

3. Bounded-truth-table reducibility vs. many-one reducibility.

THEOREM 1. *There exist two recursively enumerable, non-recursive sets A and B such that $A \leqq_{btt} B$ but not $A \leqq_m B$.*

PROOF. We choose a creative set C and take B to be the corresponding set S_1 of Post [1, p. 304] and A to be B^2. From the remark, $B^2 \leqq_{btt} B$.

Post shows that $B(=S_1)$ has the following properties:

(i) B is simple (in particular, B is not creative).

(ii) For any $n \in N$, $n \in C \equiv 2^n+1 \in B \;\&\; 2^n+2 \in B \;\&\; \cdots \;\&\; 2^{n+1} \in B$. Now suppose $B^2 \leqq_m B$, i.e., suppose there is a recursive function $g(x, y)$ of two variables such that $\langle x, y\rangle \in B^2 \equiv x \in B \;\&\; y \in B \equiv g(x, y) \in B$. Then we can reduce C many-one to B as follows: for any $n \in N$,

$$\begin{aligned} n \in C &\equiv 2^n + 1 \in B \;\&\; 2^n + 2 \in B \;\&\; 2^n + 3 \in B \;\&\; \cdots \;\&\; 2^{n+1} \in B \\ &\equiv g(2^n + 1, 2^n + 2) \in B \;\&\; 2^n + 3 \in B \;\&\; \cdots \;\&\; 2^{n+1} \in B \\ &\equiv g(g(2^n + 1, 2^n + 2), 2^n + 3) \in B \;\&\; \cdots \;\&\; 2^{n+1} \in B \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\equiv g(g(\cdots g(g(2^n + 1, 2^n + 2)2^n + 3), \cdots, 2^{n+1} - 1), 2^{n+1}) \in B. \end{aligned}$$

We define $f(n)$ to be the recursive function which, given n, applies $g(x, y)$ 2^n-1 times as indicated above. Then $n \in C \equiv f(n) \in B$. By a well-known result [2, p. 500; 3, p. 100] B must then be creative, contradicting the fact that B is simple. Therefore $B^2 \nleqq_m B$.

THEOREM 2. *There exists a simple set B such that the sets B, B^2, B^3, $\cdots$ lie in different many-one degrees of unsolvability.*

PROOF. Take B as in Theorem 1. Choose positive integers m and k and consider B^m and B^{m+k}. Clearly, $B^m \leqq_m B^{m+1} \leqq_m B^{m+k}$. Suppose B^{m+k} and B^m are in the same many-one degree. Then B^{m+1} is also in that degree and, in particular, $B^{m+1} \leqq_m B^m$. Therefore, there exists a recursive mapping h from N^{m+1} into N^m such that if h takes $\langle x_1, x_2, \cdots, x_{m+1}\rangle$ into $\langle y_1, y_2, \cdots, y_m\rangle$, then $\langle x_1, x_2, \cdots, x_{m+1}\rangle \in B^{m+1} \equiv \langle y_1, y_2, \cdots, y_m\rangle \in B^m$, i.e., $x_1 \in B \;\&\; x_2 \in B \;\&\; \cdots \;\&\; x_{m+1} \in B \equiv y_1 \in B \;\&\; y_2 \in B \;\&\; \cdots \;\&\; y_m \in B$. Again, we know that $n \in C \equiv 2^n+1 \in B \;\&\; 2^n+2 \in B \;\&\; \cdots \;\&\; 2^{n+1} \in B$, and by applying h 2^n-m times we have an effective process giving outputs $z_1(n), z_2(n), \cdots, z_m(n)$ such that $n \in C \equiv z_1(n) \in B \;\&\; z_2(n) \in B \;\&\; \cdots \;\&\; z_m(n) \in B \equiv \langle z_1(n), z_2(n), \cdots, z_m(n)\rangle \in B^m$. (If $2^n < m$, then $n \in C \equiv \langle 2^n+1, 2^n+2, \cdots, 2^{n+1}, b, b, \cdots, b\rangle \in B^m$ where b is some fixed member of B occurring $m-2^n$ times in the m-tuple.) Therefore, B^m is creative. This, however, is impossible, for $B^m \leqq_{btt} B$, and by a well-known result of Post [1, p. 304] no creative set is bounded-truth-table reducible to a simple set.

COROLLARY. *There exists at least one bounded-truth-table degree of unsolvability for recursively enumerable sets which is partitioned into* $\aleph_0$ *many-one degrees of unsolvability.*

PROOF. The corollary follows directly from Theorem 2 and from the Remark.

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