# RATIONALLY INJECTIVE MODULES FOR ALGEBRAIC LINEAR GROUPS[1]

G. HOCHSCHILD

1. Let $G$ be an algebraic linear group over a field $F$. If $G$ acts by linear automorphisms on some vector space $M$ over $F$ we say that $M$ is a *rational $G$-module* if it is the sum of finite-dimensional $G$-stable subspaces $V$ such that the representation of $G$ on each $V$ is a rational representation of $G$. The rational $G$-module $M$ is said to be *rationally injective* if, whenever $U$ is a rational $G$-module and $\phi$ is a $G$-module homomorphism of a $G$-submodule $V$ of $U$ into $M$, $\phi$ can be extended to a $G$-module homomorphism of $U$ into $M$. This notion is basic for the cohomology theory of algebraic linear groups, as developed in [1].

If $K$ is a normal algebraic subgroup of $G$ it is of interest to secure the spectral sequence relation between the cohomologies of $G$, $K$ and $G/K$. In order to do this, one needs to know that a rationally injective rational $G$-module is cohomologically trivial as a $K$-module. Although this remains unsettled over arbitrary fields, we shall see here that an even stronger result holds over fields of characteristic 0. We shall also obtain convenient criteria for injectivity that are valid over arbitrary fields.

2. Let $G$ be an algebraic linear group over an arbitrary field $F$, and let $M$ be a rationally injective rational $G$-module. It is known [1, Proposition 2.2] that, for every rational $G$-module $A$, the tensor product $A\otimes M$ is rationally injective. Invoking the rational cohomology groups as defined in [1], we easily obtain the following strong converse of this result.

PROPOSITION 2.1. *Let $G$ be an algebraic linear group over an arbitrary field $F$, and let $M$ be a rational $G$-module. Suppose that, for every finite-dimensional rational $G$-module $U$, the* 1*-dimensional rational cohomology group $H^1(G, U\otimes M)$ is* (0). *Then $M$ is rationally injective.*

PROOF. Let $A$ be a rational $G$-module, and let $\beta$ be a $G$-module homomorphism of a submodule $B$ of $A$ into $M$. We must show that $\beta$ can be extended to a $G$-module homomorphism $\alpha$ of $A$ into $M$. We apply Zorn's Lemma to the family of pairs $(C, \gamma)$, where $C$ is a $G$-module such that $B\subset C\subset A$ and $\gamma$ is a $G$-module homomorphism of $C$ into $M$ that extends $\beta$. If $(C, \gamma)$ is a maximal such pair we must

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therefore show that $C=A$. Suppose that $C\neq A$, and let $a$ be an element of $A$ that is not contained in $C$. Let $C'$ be the $G$-submodule of $A$ that is generated by $C$ and $a$. Since $A$, as a rational $G$-module, is locally finite, $C'/C$ is finite-dimensional. Thus we see that it suffices to prove the existence of an extension $\alpha$ of $\beta$ in the case where $A/B$ is finite-dimensional.

In that case, let us consider the standard exact sequence

$$\mathrm{Hom}_G(A, M) \to \mathrm{Hom}_G(B, M) \to \mathrm{Ext}_G^1(A/B, M),$$

where $\mathrm{Ext}_G^1$ denotes the rational Ext functor, in the sense of [**1**, §2]. By [**1**, §5], $\mathrm{Ext}_G^1(A/B, M)$ may, since $A/B$ is finite-dimensional, be identified with $H^1(G, (A/B)^*\otimes M)$, where $(A/B)^*$ denotes the dual of $A/B$. Thus the assumption of our proposition implies that $\mathrm{Ext}_G^1(A/B, M)=(0)$. Hence the exactness of the sequence written above shows that the restriction map $\mathrm{Hom}_G(A, M)\to\mathrm{Hom}_G(B, M)$ is surjective, which completes the proof.

For unipotent groups, our criterion simplifies to the following.

PROPOSITION 2.2. *Let $N$ be a unipotent algebraic linear group over an arbitrary field $F$. Let $M$ be a rational $N$-module such that $H^1(N, M)=(0)$. Then $M$ is rationally injective.*

PROOF. By Proposition 2.1, it suffices to show that $H^1(N, U\otimes M)=(0)$, for every finite-dimensional rational $N$-module $U$. Define $U_1$ to be the submodule $U^N$ of the $N$-fixed elements of $U$. If $U_p$ is already defined, define $U_{p+1}$ as the inverse image in $U$ of $(U/U_p)^N$. Since $N$ is unipotent, the rational representation of $N$ on $U$ is unipotent, whence $U$ is the union of the submodules $U_p$.

Since the representation of $N$ on $U_1$ is trivial, we have evidently $H^1(N, U_1\otimes M)=U_1\otimes H^1(N, M)=(0)$. Similarly,

$$H^1(N, (U_{p+1}/U_p)\otimes M) = (0).$$

From the exact sequence $(0)\to U_p\otimes M\to U_{p+1}\otimes M\to(U_{p+1}/U_p)\otimes M\to(0)$, we get the exact sequence

$$H^1(N, U_p\otimes M)\to H^1(N, U_{p+1}\otimes M)\to H^1(N, (U_{p+1}/U_p)\otimes M) = (0).$$

Hence we see by induction on $p$ that $H^1(N, U_p\otimes M)=(0)$, for all $p$. Since $U=U_p$, for some $p$, we conclude that $H^1(N, U\otimes M)=(0)$, q.e.d.

3. Now we shall assume that the base field $F$ is of characteristic 0. In that case, the unipotent groups are decisive for the rational cohomology (cf. [**1**]). First we note a special case of the result to be proved.

LEMMA 3.1. *Let $G$ be an algebraic linear group over the field $F$ of characteristic* 0, *and let $L$ be a unipotent normal algebraic subgroup of $G$. Let $M$ be a rationally injective rational $G$-module. Then $M$ is rationally injective also as an $L$-module.*

PROOF. By [1, Theorem 3.1], there is a rational representative map $\rho: G \to L$ such that $\rho(yx) = y\rho(x)$, for every $x \in G$ and every $y \in L$. Hence we may apply [1, Proposition 2.2] to conclude that $M$ is rationally injective also as an $L$-module.

Now we are ready to prove the main result.

THEOREM 3.1. *Let $G$ be an algebraic linear group over a field of characteristic* 0, *let $K$ be a normal algebraic subgroup of $G$, and let $M$ be a rationally injective rational $G$-module. Then $M$ is rationally injective also as a $K$-module.*

PROOF. Let $N$ be the maximum normal unipotent subgroup of $K$. Then $N$ is a unipotent normal algebraic subgroup of $G$. Hence, by Lemma 3.1, $M$ is rationally injective as an $N$-module. Let $U$ be any rational $K$-module. By [1, Proposition 2.2], $U \otimes M$ is rationally injective as an $N$-module. By [1, Theorem 5.2], the cohomology restriction map sends $H(K, A)$ isomorphically onto the $(K/N)$-fixed part $H(N, A)^{K/N}$ of $H(N, A)$, for every rational $K$-module $A$. In particular, it follows that $H^1(K, U \otimes M) = (0)$. By Proposition 2.1, this implies that $M$ is rationally injective as a $K$-module.

4. *A rational module may be cohomologically trivial without being rationally injective.* Let $F$ be a field of characteristic 0, let $F^+$ denote the additive group of $F$, $F^*$ the multiplicative group of $F$, $G$ the direct product $F^+ \times F^*$. This group $G$ may be regarded as an algebraic linear group over $F$ in a natural way so that $F^+$ is the maximum normal unipotent subgroup of $G$. Let $M$ be the rational $G$-module whose underlying vector space is $F$ and on which $F^+$ acts trivially while $F^*$ acts by multiplication. One sees easily from the explicit cochain description of the cohomology group [1, §2] that $H^1(F^+, M) = F$ and $H^n(F^+, M) = (0)$, for all $n > 1$. Furthermore, we have $(H^1(F^+, M))^G = M^G = (0)$. Now it follows from [1, Theorem 5.2] (as quoted in the proof of Theorem 3.1) that $H^n(G, M) = (0)$, for all $n > 0$, i.e., that $M$ is cohomologically trivial. On the other hand, $M$ is not rationally injective. For otherwise, by Lemma 3.1, $M$ would be rationally injective as an $F^+$-module, which would contradict our result that $H^1(F^+, M) = F \neq (0)$.

Finally, we observe that, *in Theorem* 3.1, *the condition that $K$ be normal in $G$ is not superfluous.* Let $G$ be a fully reducible algebraic linear group over a field $F$ of characteristic 0, e.g. the group of all invertible 2 by 2 matrices, and let $K$ be a nontrivial unipotent algebraic subgroup of $G$, e.g. the group of the matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}; \quad a \in F.$$

Since $G$ is fully reducible and $F$ is of characteristic 0, every rational $G$-module is semisimple, from which it follows that every rational $G$-module is rationally injective. In particular, the rational $G$-module $F$, with $G$ acting trivially, is rationally injective. But $F$ is not rationally injective as a $K$-module.

## Reference

1. G. Hochschild, *Cohomology of algebraic linear groups*, Illinois J. Math. **5** (1961), 492–519.

University of California, Berkeley