RATIONAL NORMAL MATRICES SATISFYING THE INCIDENCE EQUATION

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1. **Introduction.** The construction of a finite projective plane with n points on a line is equivalent to determining an integral $v \times v$ matrix A $(v = n^2 + n + 1)$ satisfying

$$(1) \qquad A\,{}^tA = {}^tAA = nI + S$$

where S is the matrix all of whose coordinates are 1. The Bruck-Ryser Theorem [2] asserts that when $n \equiv 1$ or 2 (mod 4), a necessary and sufficient condition for the existence of a rational A satisfying (1) is that n be a sum of two squares. In [1] Albert gives a construction for such a rational A. The purpose of this note is to give a simpler construction.

2. **Notation.** We denote I_r the $r \times r$ identity matrix, S_r the $r \times r$ matrix all of whose entries are 1, e_r the $1 \times r$ matrix (row vector) all of whose entries are 1. If $m = \frac{1}{2}(v-1) = \frac{1}{2}n(n+1)$, we denote

$$E = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

so that $E\,{}^tE = I_{2m}$ and $E + {}^tE = 0$. We suppose $n = a^2 + b^2$ and let $H = aI_{2m} + bE$ so that $H\,{}^tH = nI_{2m}$. Let

$$P = \begin{bmatrix} 0 & \dfrac{1}{n}\, e_{2m}H \\ {}^te_{2m} & H \end{bmatrix};$$

then clearly P is a rational $v \times v$ matrix satisfying $P\,{}^tP = nI_v + S_v$. Moreover

$${}^tPP = nI_v + {}^txx \quad \text{where} \quad x = \left(n \;\; \frac{1}{n}\, e_{2m}H \right)$$

is a $1 \times v$ row vector.

3. **A rational solution.** To obtain a rational solution of (1) it suffices to find a rational orthogonal $v \times v$ matrix T such that if $A = PT$ then ${}^tAA = nI_v + S_v$. But ${}^tAA = {}^tT\,{}^tPPT = {}^tT(nI_v + {}^txx)T = nI_v + {}^t(xT)(xT)$;

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hence it suffices to find a rational orthogonal T satisfying ${}^t(xT)(xT) = S_v$. Since $S_v = {}^te_ve_v$, we must find a rational T satisfying $xT = e_v$. Noting that $x{}^tx = e_v{}^te_v = v$, i.e., the vectors x and e_v have the same length, it is apparent that the symmetry with respect to the plane perpendicular to the vector $x - e_v$ is the required transformation:

$$T = I_v - \frac{2{}^t(x - e)(x - e)}{(x - e){}^t(x - e)}.$$

References

1. A. A. Albert, *Rational normal matrices satisfying the incidence equation*, Proc. Amer. Math. Soc. 4 (1953), 554–559.

2. R. H. Bruck and H. J. Ryser, *The nonexistence of certain finite projective planes*, Canad. J. Math. 1 (1949), 88–93.

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