A NOTE ON AN INEQUALITY OF M. MARCUS AND M. NEWMAN

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The permanent of an n-square matrix $A=(a_{ij})$ is denoted by $p(A)$ and is defined by

$$p(A) = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1}$$

where S_n is the symmetric group of degree n. A real matrix S is said to be doubly stochastic if its entries are nonnegative and its row sums and column sums are all 1.

In a recent paper [2] Marcus and Newman proved that if A is a symmetric positive semi-definite doubly stochastic matrix then

$$p(A) \geqq n!/n^n \tag{2}$$

with equality if and only if $A=J_n$, the matrix all of whose entries are $1/n$. This result is a partial answer to a conjecture of van der Waerden [3] stating that if S is any doubly stochastic matrix then $p(S) \geqq p(J_n)=n!/n^n$ with equality if and only if $S=J_n$.

In the present note we extend the result of Marcus and Newman to all positive semi-definite hermitian matrices which have $e=(1, 1, \cdots, 1)$ as a characteristic vector and prove it by a new method. We first establish a lemma which is a weakened version of Theorem 2 in [1] and use it in conjunction with the following inequality also due to Marcus and Newman (Theorem 5 in [2]).

If A and B are complex n-square matrices then

$$|p(AB)|^2 \leqq p(AA^*)p(B^*B) \tag{3}$$

with equality if and only if either (a) a row of A or a column of B is zero, or (b) there exist a diagonal matrix G and a permutation matrix P such that $A^*=BGP$.

Let Φ_n denote the set of positive semi-definite hermitian n-square matrices which have e as a characteristic vector. In other words, Φ_n contains all nonnegative multiples of positive semi-definite hermitian n-square matrices whose row sums are all 1. Thus if $H\in\Phi_n$ then $HJ_n=\lambda_1 J_n$ for some nonnegative λ_1.

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LEMMA. *For any H in Φ_n there exists a matrix B in Φ_n such that $B^2=H$.*

PROOF. Let all the row sums of H be equal to $\lambda_1 \geqq 0$. Then λ_1 is a characteristic root of H and e is a corresponding characteristic vector. There exists therefore a unitary matrix U with $e/\sqrt{n}$ as its first column vector and such that

$$U^*HU = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the nonnegative characteristic roots of H. Let $D=\operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n})$ and let $B=UDU^*$. Clearly B is positive semi-definite hermitian and $B^2=H$. It remains to prove that e is a characteristic vector of B. Let $X^{(1)}$ denote the first column vector of a matrix X. Then

$$U^*e = U^*\sqrt{n}U^{(1)} = \sqrt{n}(U^*U)^{(1)} = \begin{pmatrix} \sqrt{n} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

Hence $XU^*e=\sqrt{n}X^{(1)}$ for any n-square matrix X. In particular

$$Be = UDU^*e = \sqrt{n}(UD)^{(1)} = \sqrt{n}UD^{(1)} = \sqrt{(\lambda_1 n)}U^{(1)} = \sqrt{\lambda_1}e.$$

THEOREM. *If $H \in \Phi_n$ and the row sums of H are all equal to λ_1 then*

$$p(H) \geqq n!(\lambda_1/n)^n$$

with equality if and only if either a row of H is zero or H is a nonnegative multiple of J_n.

PROOF. By the lemma, $H=B^2$ where $B \in \Phi_n$ and therefore $B=B^*$ and $BJ_n=\sqrt{\lambda_1}J_n$. Hence, by (3),

$$| p(BJ_n) |^2 \leqq p(BB^*)p(J_n^*J_n),$$

i.e.,

$$(p(\sqrt{\lambda_1}J_n))^2 \leqq p(H)p(J_n)$$

and therefore

$$p(H) \geqq \lambda_1^n p(J_n) = n!(\lambda_1/n)^n.$$

Equality will occur (see conditions for equality in (3)) if and only if either (a) a row of B, and therefore a row of H, is zero, or (b) there exist a diagonal matrix G and a permutation matrix P such that

$B=J_nGP$. In case (b), $H=BB^*=J_nG^2J_n$ and clearly $J_nG^2J_n = (\mathrm{tr}(G^2)/n)J_n$.

COROLLARY. *If S is a symmetric positive semi-definite doubly stochastic n-square matrix then*

$$p(S) \geqq n!/n^n$$

with equality if and only if $S=J_n$.

REFERENCES

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