

A REMARK ON UNIVERSAL SIGMA INTEGRABILITY

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Our purpose is to answer the question of Leader [1, p. 234], by producing an example of a sigma algebra σ of subsets of a set X and a *universally σ -integrable* function f (i.e., $\int f d\mu$ exists for every countably additive set function μ on σ) which is not measurable with respect to σ .

We shall show that the sigma algebra σ of Borel subsets of the interval $[0, 1]$ admits nonmeasurable universally σ -integrable functions (R. E. Zink suggested that this Borel algebra might yield an example).

Lusin has shown [2] that, subject to the continuum hypothesis, there exists an uncountable subset V of $X = [0, 1]$ such that each perfect nowhere dense subset G of X contains at most an enumerable set of points of V . Let H be a subset of V that is not a Borel set. We shall now show that the characteristic function $\chi(H)$ of H is universally sigma integrable. To this end suppose u is a countably additive set function on σ . Then [5] $u = c + j$ where c is continuous and $j = \sum j_k$ where j_k is a two valued jump function, the range of k is at most enumerable, and the variation of j is the sum of the variations of the j_k 's; moreover, since the variation of u is the variation of j plus the variation \bar{c} of c , each of j , c , and \bar{c} is countably additive on σ . Corresponding to each j_k there is [4, Theorem 27.1] a point x_k such that $j_k(E) = j_k(E \cap [x_k])$ for every Borel subset E of X . Hence $\int \chi(H) dj = \sum j_k(H \cap [x_k])$ and it suffices to show that there exists a Borel subset B of X such that $H \subset B$ and $\bar{c}(B) = 0$ (then $\int \chi(H) dc = 0$). There exists [3] a set $K = \bigcup_i F_i$, F_i closed for $i \geq 1$, of the first category in X and a set L such that $\bar{c}(L) = 0$ and $K \cup L = X$. For each positive integer i there exists a perfect set G_i and an at most enumerable set H_i such that $F_i = G_i \cup H_i$. Thus $H \cap K = H \cap (\bigcup_i F_i) = H \cap (\bigcup_i (G_i \cup H_i)) = \bigcup_i (H \cap (G_i \cup H_i)) \subset \bigcup_i ((H \cap G_i) \cup H_i)$ is at most enumerable since every $H \cap G_i$ (F_i is a closed subset of X and K is of the first category in X ; hence F_i is nowhere dense in X which implies that G_i is a perfect nowhere dense subset of X) and every H_i is at most enumerable. Since \bar{c} is countably additive on σ and \bar{c} vanishes on one point sets, \bar{c} vanishes on countable sets. Hence $H = H \cap (K \cup L) \subset (H \cap K) \cup L$ and $\bar{c}(H \cap K) \cup L \leq \bar{c}(H \cap K) + \bar{c}(L) = 0$.

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