ON THE HYPERBOLIC CAPACITY AND CONFORMAL MAPPING[1]

CH. POMMERENKE

1. Let E be a compact set in $D=\{|z|<1\}$. Tsuji [6] has introduced the hyperbolic capacity of E which can be defined by

$$\text{caph } E = \lim_{n\to\infty} \max_{z_0,\ldots,z_n\in E} \prod_{\mu=0}^{n} \prod_{\nu\neq\mu} \left| \frac{z_\mu - z_\nu}{1 - \bar{z}_\nu z_\mu} \right|^{1/n(n+1)}. \tag{1}$$

Also,

$$\min_f \max_{z\in E} |f(z)|^{1/n} \to \text{caph } E \tag{2}$$

as $n\to\infty$ where the minimum is taken over all functions

$$f(z) = \prod_{\nu=1}^{n} e^{i\alpha_\nu}(z - z_\nu)/(1 - \bar{z}_\nu z) \qquad (\alpha_\nu \text{ real}, \ |z_\nu| < 1). \tag{3}$$

We shall first obtain another formula for caph E. Leja [1] has proved an analogous formula for the capacity of a plane set.

LEMMA. *Let E be a compact set in D. For each $n=1, 2, \cdots$ choose $n+1$ points $z_0, \cdots, z_n$ in E such that*

$$\prod_{\mu=0}^{n} \prod_{\nu\neq\mu} |z_\mu - z_\nu| / |1 - \bar{z}_\nu z_\mu|$$

becomes maximal. Numerate these points so that

$$\begin{aligned} A_n &= \prod_{\nu=1}^{n} |z_0 - z_\nu| / |1 - \bar{z}_\nu z_0| \\ &= \min_\mu \prod_{\nu\neq\mu} |z_\mu - z_\nu| / |1 - \bar{z}_\nu z_\mu| . \end{aligned} \tag{4}$$

If

$$f_n(z) = \prod_{\nu=1}^{n} \frac{1 - \bar{z}_\nu}{1 - z_\nu} \frac{z - z_\nu}{1 - \bar{z}_\nu z} \tag{5}$$

then

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$$\max_{z\in E} |f_n(z)| = A_n, \tag{6}$$

and, as $n\to\infty$

$$A_n^{1/n} \to \text{caph } E.$$

PROOF. For $|\zeta_1|<1$, $|\zeta_2|<1$ we write $[\zeta_1, \zeta_2] = |\zeta_1-\zeta_2|/|1-\bar{\zeta}_2\zeta_1|$. Let $z\in E$. Comparing the system $z, z_1, \cdots, z_n$ of points in E with the maximal system $z_0, z_1, \cdots, z_n$ we see that

$$\begin{vmatrix} 1\cdot[z, z_1] & \cdots & [z, z_n] \\ [z_1, z]\cdot 1 & \cdots & [z_1, z_n] \\ \vdots & & \vdots \\ [z_n, z][z_n, z_1] & \cdots & 1 \end{vmatrix} \leqq \begin{vmatrix} 1\cdot[z_0, z_1] & \cdots & [z_0, z_n] \\ [z_1, z_0]\cdot 1 & \cdots & [z_1, z_n] \\ \vdots & & \vdots \\ [z_n, z_0][z_n, z_1] & \cdots & 1 \end{vmatrix}.$$

Hence $|f_n(z)| \leqq A_n$, with equality for $z=z_0$, which proves (6). Since f_n has the form (3) it follows that $\min_f \max_{z\in E} |f(z)| \leqq A_n$. Therefore by (2)

$$\liminf_{n\to\infty} A_n^{1/n} \geqq \text{caph } E.$$

On the other hand, (4) implies

$$A_n^{n+1} \leqq \prod_{\mu=0}^{n} \prod_{\nu\neq\mu} [z_\mu, z_\nu].$$

Hence (1) shows that $\limsup_{n\to\infty} A_n^{1/n} \leqq \text{caph } E$, and the Lemma follows.

2. Let E be a compact set in $D=\{|z|<1\}$. Then $D\backslash E$ is an open set of which exactly one component region G has $\{|z|=1\}$ as part of the boundary. I shall give an elementary proof of the following theorem.

THEOREM 1. *Let* $\rho = \text{caph } E > 0$. *If* $f_n(z)$ *is defined by* (5) *then*

$$g(z) = \lim_{n\to\infty} f_n(z)^{1/n} \tag{7}$$

exists locally uniformly in $H = G\cup\{1\leqq |z| < r\}$ *for some* $r>1$, *and* $g(z)$ *is the smallest function satisfying*

(a) $g(z)$ *is locally analytic*[2] *and of single-valued modulus in* H,

(b) $|g(z)|=1$ *for* $|z|=1$,

(c) $1\geqq |g(z)| \geqq \rho$ *for* $z\in G$,

that is, if $h(z)$ *also satisfies these three conditions then* $|g(z)| \leqq |h(z)|$ *for* $z\in G$.

[2] This means that $g(z)$ is analytic on the universal covering surface of H.

Furthermore, $g(1)=1$ *and*

$$\int_0^{2\pi} d \arg g(e^{i\theta}) = 2\pi. \tag{8}$$

If ζ *is a boundary point of* G *that lies on a continuum contained in* E *then* $|g(z)|\to\rho$ *for* $z\to\zeta$, $z\in G$.

Finally, if E *is a continuum then* $\rho>0$, *and* $w=g(z)$ *maps* G *conformally and one-to-one onto* $\{\rho<|w|<1\}$.

REMARKS. Let

$$\omega(z) = \log(\rho^{-1}|g(z)|)/\log \rho^{-1}.$$

Then Theorem 1 shows that $\omega(z)$ is the smallest function satisfying

(a′) $\omega(z)$ is single-valued and harmonic in H,

(b′) $\omega(z)=1$ for $|z|=1$,

(c′) $1\geqq\omega(z)\geqq 0$ for $z\in G$,

that is, if $\nu(z)$ also satisfies these three conditions then $\omega(z)\leqq\nu(z)$. If the boundary of G consists of a finite number of nondegenerate continua then $\omega(z)=0$ on the boundary points of G that lie in D. Hence $\omega(z)$ is the harmonic measure of $\{|z|=1\}$ with respect to G. By (8),

$$1/\log \rho^{-1} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial r}\,\omega(re^{i\theta})\Big|_{r=1} d\theta. \tag{9}$$

Of course, we could have started with the harmonic measure and then proved (9). But the method applied here is simpler and more constructive. It does not use set-functions, the solvability of the Dirichlet problem or the Riemann mapping theorem. The existence of a function that maps a doubly-connected region onto an annulus is established (see also [**4**]).

The following proof uses (with some simplifications) the method of extremal points developed by Leja [**1**; **2**; **3**].

PROOF. a. The Lagrange interpolation formula shows that

$$\sum_{\mu=0}^{n}\left(\prod_{\nu\neq\mu}\frac{z-z_\nu}{z_\mu-z_\nu}\cdot\prod_{\nu=1}^{n}(1-\bar z_\nu z_\mu)\right)=\prod_{\nu=1}^{n}(1-\bar z_\nu z).$$

Hence

$$\max_\mu\left(\prod_{\nu\neq\mu}\left|\frac{z-z_\nu}{z_\mu-z_\nu}\right|\cdot\prod_{\nu=1}^{n}\left|\frac{1-\bar z_\nu z_\mu}{1-\bar z_\nu z}\right|\right)\geqq\frac{1}{n+1}\cdot$$

Let $q(z)=\min_{\zeta_1,\zeta_2\in E}|z-\zeta_1|/|z-\zeta_2|$ (for $z\in G$). Since $E\subset\{|z|\leqq a\}$ for some $a<1$ it follows that

$$\max_{\mu}\left(\prod_{\nu=1}^{n}\left|\frac{z-z_\nu}{1-\bar{z}_\nu z}\right|\cdot\prod_{\nu\neq\mu}\left|\frac{1-\bar{z}_\nu z_\mu}{z_\mu-z_\nu}\right|\right)\geqq\frac{(1-a^2)q(z)}{2(n+1)},$$

and because of (4)

$$|f_n(z)| \geqq \frac{(1-a^2)q(z)}{2(n+1)}A_n. \tag{10}$$

We put $r=2/(1+a)>1$. Since $|z-z_\nu|/|1-\bar{z}_\nu z|\leqq(r+a)/(1-ar)$ $<4/(1-a)$ for $|z|\leqq r$, (5) shows that

$$|f_n(z)|^{1/n}<4/(1-a) \qquad (|z|\leqq r). \tag{11}$$

b. Let $H=G\cup\{1\leqq|z|\leqq r\}$ and $g_n(z)=f_n(z)^{1/n}$. The functions $g_n(z)$ are locally analytic in H, and $|g_n(z)|$ is single-valued. By (11) and Montel's theorem we can find a sequence n_k such that $g_{n_k}(z)$ converges locally uniformly in H. Let $g(z)$ be the limit function. Since by the Lemma $A_n^{1/n}\to\rho$, inequality (10) implies $|g(z)|\geqq\rho$. Also $|g(z)|=1$ for $|z|=1$ so that $g(z)$ satisfies (a), (b) and (c).

Let $h(z)$ be any function satisfying these three conditions, and let z^* be a point in G. Given $\epsilon>0$ we choose a fixed k so large that $|g_{n_k}(z^*)|>e^{-\epsilon}|g(z^*)|$. Since $\rho>0$ we can take k so that also $A_{n_k}^{1/n_k}<\rho e^{\epsilon}$. Then it follows from (6) that $|g_{n_k}(z)|\leqq\rho e^{\epsilon}$ for $z\in E$. We choose analytic curves in G so near to E that their union C separates E from z^* and from $\{|z|=1\}$, and that $|g_{n_k}(z)|\leqq\rho e^{2\epsilon}$ for $z\in C$. Because $|h(z)|$ $\geqq\rho$ for $z\in G$,

$$|g_{n_k}(z)|/|h(z)| \leqq \rho e^{2\epsilon}/\rho = e^{2\epsilon}$$

for $z\in C$. Since the left side is $=1$ for $|z|=1$ it follows from the maximum principle that the inequality holds also for $z=z^*$. Hence

$$|g(z^*)| < e^{\epsilon}|g_{n_k}(z^*)| \leqq e^{3\epsilon}|h(z^*)|$$

for every $\epsilon>0$ and therefore $|g(z^*)|\leqq|h(z^*)|$.

Since $f_n(1)=1$ we obtain $g(1)=1$, and by the argument principle

$$\int_0^{2\pi} d\arg g_n(e^{i\theta}) = \frac{1}{n}\int_0^{2\pi} d\arg f_n(e^{i\theta}) = 2\pi \tag{12}$$

from which (8) follows.

If $g_n(z)$ did not converge there would be a limit function $h(z)\not\equiv g(z)$ for some other subsequence of g_n as Montel's theorem shows. From what we have proved it follows that $|h(z)|\geqq|g(z)|$. Reversing the roles of g and h we also get $|g(z)|\geqq|h(z)|$. Hence $|h(z)|=|g(z)|$, and $g(1)=h(1)=1$ implies $g(z)\equiv h(z)$. Therefore $g_n(z)\to g(z)$ as $n\to\infty$.

c. We assume now that E is a continuum. We do not know yet that

$\rho>0$. If $\rho=0$ then $\lim g_n(z)$ might not exist. In this case let $g(z)$ be the limit function in H for some convergent subsequence of g_n which exists by (11). The region H is doubly connected, and every simply closed curve in H is homotopic either to a point or the unit circle. Therefore the functions $g(z)$ and $g_n(z)$ are single-valued because of (8) and (12). Let c be any point with $\rho<|c|<1$. It follows from (12) that $w=g_n(z)$ maps $\{|z|=1\}$ one-to-one onto $\{|w|=1\}$. Hence

$$\text{(13)}\qquad \frac{1}{2\pi i}\int_{|z|=1}\frac{g_n'(z)}{g_n(z)-c}\,dz=\frac{1}{2\pi i}\int_{|w|=1}\frac{1}{w-c}\,dw=1.$$

We choose analytic curves C_m $(m=1,2,\cdots)$ enclosing E so that the regions between C_m and $\{|z|=1\}$ approach G. By the Lemma we can choose them so that $|g_n(z)|<|c|$ on each C_m for sufficiently large n. Then

$$\text{(14)}\qquad \frac{1}{2\pi i}\int_{C_m}\frac{g_n'(z)}{g_n(z)-c}\,dz=0.$$

Making $n\to\infty$ we obtain from (13) and (14) that

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{g'(z)}{g(z)-c}\,dz-\frac{1}{2\pi i}\int_{C_m}\frac{g'(z)}{g(z)-c}\,dz=1$$

for all m. Hence $g(z)$ assumes the value c exactly once in G. Therefore $w=g(z)$ maps G one-to-one onto $\{\rho<|w|<1\}$.

Suppose that $\rho=0$ were true. Then the inverse function $\psi(w)$ of $w=g(z)$ would be analytic and univalent in $\{0<|w|\leqq 1\}$. Since $|\psi(w)|<1$ it would follow that $\psi(w)$ is bounded and univalent in $\{|w|\leqq 1\}$. Since $|\psi(w)|=1$ for $|w|=1$ this would imply that $\psi(w)$ is a linear function and therefore E a point.

d. Let again E be arbitrary and let ζ be a boundary point lying on a continuum B that is contained in E. Let $g_0(z)$ be the function that maps the doubly-connected region between B and $\{|z|=1\}$ onto $\{\rho_0<|w|<1\}$. Let λ be such that $\rho_0^\lambda=\rho$. Then $h(z)=g_0(z)^\lambda$ satisfies the conditions (a), (b) and (c) of Theorem 1. Hence, as we have already proved, $\rho\leqq|g(z)|\leqq|h(z)|=|g_0(z)|^\lambda$. Since for simple topological reasons $|g_0(z)|\to\rho_0$ as $z\to\zeta$, $z\in G$ it follows that $|g(z)|\to\rho$.

3. We shall now prove an analogue to a result by Walsh [7] about the ordinary Green's function. We introduce the hyperbolic metric in the unit disk D. A circle perpendicular to $\{|z|=1\}$ will be called a geodesic.

Theorem 2. *Let E be a compact set in D with* caph $E>0$, *and let*

$g(z)$ be the function defined in Theorem 1. *Let* $L(r)=\{z: |g(z)|=r\}$ (caph $E<r<1$). *Then at every point of $L(r)$ the inner geodesic normal to $L(r)$ intersects the hyperbolically convex hull K of E.*

REMARKS. The hyperbolically convex hull of E is defined as the smallest closed set $K\supset E$ that is convex in the hyperbolic metric in the sense that together with any two points also the geodesic segment between these two points belongs to K. The set $L(r)$ is the union of a finite number of closed analytic curves which may have multiple points though. At the multiple points $g'(z)$ vanishes. It is easy to see that all multiple points lie in K (see [8, p. 157]).

PROOF. With the notations of Theorem 1 let $L_n(r)=\{z: |f_n(z)|=r^n\}$. We first prove that the inner geodesic normal to $L_n(r)$ at any $\zeta\in L_n(r)$ intersects K. Suppose this were false. Then $\zeta\notin K$. By a conformal mapping of the unit disk onto itself we can make $\zeta=0$. Then the inner geodesic normal becomes a straight ray and is separated from K by a line. We may thus assume that $K\subset\{\operatorname{Re} z<0\}$ and that the inner normal lies in $\{\operatorname{Re} z\geqq 0\}$. Writing $z_\nu=x_\nu+iy_\nu$ we have $x_\nu<0$. Hence

$$\frac{d}{dz}\log f_n(z)\bigg|_{z=0}=\sum_{\nu=1}^{n}\frac{1-|z_\nu|^2}{(z-z_\nu)(1-\bar{z}_\nu z)}\bigg|_{z=0}$$
$$=\sum_{\nu=1}^{n}\frac{1-|z_\nu|^2}{|z_\nu|^2}(-x_\nu+iy_\nu).$$

Therefore $\operatorname{Re} f_n'(0)/f_n(0)>0$, and the inner geodesic normal to $L_n(r)=\{z: \operatorname{Re}\log f_n(z)=n\log r\}$ at 0 lies in $\{\operatorname{Re} z<0\}$ (except for the point 0), in contradiction to our assumption. Theorem 2 follows because $f_n(z)^{1/n}\to g(z)$ locally uniformly in G.

4. We shall apply Theorem 2 to obtain a result about the distortion under the conformal mapping of an annulus. It is a generalization of Theorem 6 in [5]. The closure of the region inside D that lies between two geodesics with common endpoint ζ will be called a geodesic sector of vertex ζ.

THEOREM 3. *Let G be a doubly connected region in D, with $\{|z|=1\}$ as outer and E as inner boundary. Let $w=g(z)$ be the function that maps G conformally onto $\{\rho<|w|<1\}$ such that $g(1)=1$. Let S be the smallest geodesic sector of vertex* 1 *that contains E, and let T be the component of $S\backslash E$ that contains* 1. *If R is the curve that $w=g(z)$ maps onto the interval* $(\rho, 1)$ *then $R\subset T$.*

By the conformal mapping $z^*=(1+z)/(1-z)$ of D onto $\{\operatorname{Re} z^*>0\}$ we see that Theorem 3 is equivalent with

THEOREM 3*. *Let G^* be a doubly connected region in $\{\operatorname{Re} z^*>0\}$ with $\{\operatorname{Re} z^*=0\}$ and E^* as boundaries. Let $w=g^*(z)$ map G^* conformally onto $\{\rho<|w|<1\}$ such that $g^*(+\infty)=1$. Let S^* be the smallest strip parallel to the real axis that contains E^*, and let T^* be the component of $S^*\backslash E^*$ that contains $+\infty$. If R^* is the curve that $w=g^*(z)$ maps onto the interval $(\rho, 1)$ then $R^*\subset T^*$.*

PROOF. Theorem 2 shows that all tangents to R^* certainly intersect S^*. If $S^*=\{a\leqq \operatorname{Im} z^*\leqq b\}$ it follows that

$$\limsup_{z^*\in R^*,\ z^*\to+\infty} \operatorname{Im} z^* \leqq b. \tag{15}$$

Also, all accumulation points of the left end of R^* lie on E^*, hence in $\{\operatorname{Im} z^*\leqq b\}$. Suppose $\max_{z^*\in R^*} \operatorname{Im} z^*>b$. Together with (15) this would imply that the maximum is assumed, say at $z=c$. But $c\notin S^*$, and the tangent to R^* at c is parallel to S^* so that it would not intersect S^*. Thus we have shown that $\operatorname{Im} z^*\leqq b$, and also $\operatorname{Im} z^*\geqq a$, for $z^*\in S^*$. Hence $R^*\subset S^*$. Since R^* is a curve and contains points with large real part, R^* has to lie in the component T^* of $S^*\backslash E^*$ that contains the point $+\infty$.

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CORNELL UNIVERSITY