SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually
elegant and polished character, for which there is normally no other outlet.

A SIMPLE EXAMPLE OF A TRANSCENDENTAL ENTIRE
FUNCTION THAT TOGETHER WITH ALL ITS
DERIVATIVES ASSUMES ALGEBRAIC VALUES
AT ALL ALGEBRAIC POINTS

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Let \( \{z_i\} = \{z_1, z_2, z_3, \cdots \} \) be an enumeration of all algebraic numbers [1]. Construct a sequence \( \{\xi_j\} = \{\xi_1, \xi_2, \xi_3, \cdots \} \)
= \( \{z_1, z_1, z_2, z_1, z_2, z_3, z_1, \cdots \} \) so that all of the algebraic numbers appear an infinite number of times in \( \{\xi_j\} \). Then, for algebraic numbers \( a_n \) with \( 0 < |a_n| < (n! \cdot \prod_{j=1}^{\infty} (1 + |\xi_j|))^{-1} \), the function \( f(z) = \sum_{n=0}^{\infty} a_n \cdot \prod_{j=1}^{n} (z - \xi_j) \) is an entire function having the said property. Since \( |z - \xi_j| \leq 1 + |\xi_j| \) for \( |z| \leq 1 \) and \( |z - \xi_j| \leq |z| \cdot (1 + |\xi_j/z|) \)
\( |z| \cdot (1 + |\xi_j|) \) for \( |z| > 1 \), the series for \( f(z) \) converges absolutely and uniformly in \( |z| \leq R < \infty \) and \( |f(z)| \leq \max \{e, 8^{1/2}\} \). Since \( f^{(m)}(\xi_j) \)
is a polynomial of \( \xi_j \) with algebraic coefficients \( a_n \) and \( \{\xi_j\} \) contains all algebraic numbers infinitely many times, \( f^{(m)}(\xi) \) must be an algebraic
number for any algebraic number \( \xi \).

If we ask the general question: For what sets, \( S \), of complex numbers do there exist transcendental entire functions which, together
with all their derivatives, map \( S \) into \( S' \)?, we see immediately that the above construction can be applied to any dense denumerable set,
or to any denumerable ring which has 0 as a limit point, such as the
ring of rationals. A similar method can be applied to discrete infinite
rings such as the ring of integers. The question for nondenumerable
nonclosed rings \( S \) remains open.

REFERENCES

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