SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A SIMPLE EXAMPLE OF A TRANSCENDENTAL ENTIRE FUNCTION THAT TOGETHER WITH ALL ITS DERIVATIVES ASSUMES ALGEBRAIC VALUES AT ALL ALGEBRAIC POINTS

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Let $\{z_i\} = \{z_1, z_2, z_3, \cdots\}$ be an enumeration of all algebraic numbers [1]. Construct a sequence [2] $\{\zeta_j\} = \{\zeta_1, \zeta_2, \zeta_3, \cdots\} = \{z_1, z_1, z_2, z_1, z_2, z_3, z_1, \cdots\}$ so that all of the algebraic numbers appear an infinite number of times in $\{\zeta_j\}$. Then, for algebraic numbers a_n with $0 < |a_n| < (n! \cdot \prod_{j=1}^{n} (1+|\zeta_j|))^{-1}$, the function $f(z) = \sum_{n=0}^{\infty} a_n \cdot \prod_{j=1}^{n} (z-\zeta_j)$ is an entire function having the said property. Since $|z-\zeta_j| \leqq 1+|\zeta_j|$ for $|z| \leqq 1$ and $|z-\zeta_j| \leqq |z| \cdot (1+|\zeta_j/z|) < |z| \cdot (1+|\zeta_j|)$ for $|z| > 1$, the series for $f(z)$ converges absolutely and uniformly in $|z| \leqq R < \infty$ and $|f(z)| \leqq \max\{e, e^{|z|}\}$. Since $f^{(m)}(\zeta_j)$ is a polynomial of ζ_j with algebraic coefficients a_n and $\{\zeta_j\}$ contains all algebraic numbers infinitely many times, $f^{(m)}(\zeta)$ must be an algebraic number for any algebraic number ζ.

If we ask the general question: For what sets, S, of complex numbers do there exist transcendental entire functions which, together with all their derivatives, map S into S?, we see immediately that the above construction can be applied to any dense denumerable set, or to any denumerable ring which has 0 as a limit point, such as the ring of rationals. A similar method can be applied to discrete infinite rings such as the ring of integers. The question for nondenumerable nonclosed rings S remains open.

References

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