EXTENSION OF A THEOREM OF A. WINTNER

L. SUYEMOTO AND P. WALTMAN

In [1] A. Wintner established the following theorem.

THEOREM 1. *The differential equation*

$$y'' + f(x)y = 0 \tag{1}$$

cannot have an (L^2)*-solution if*

$$\int_0^\infty x^3 |f(x)|^2 dx < \infty \tag{2}$$

(*where* $\limsup_{x\to\infty} |f(x)| \neq 0$ *is allowed*).

In this note, Theorem 1 will be extended to the more general equation

$$y'' + f(x)y^p = 0. \tag{3}$$

THEOREM 2. *If* (2) *holds, and* $p \geqq 1$, *then* (3) *has no* (L^{2p})*-solutions.*

For $p=1$ this reduces to the result of Wintner. First we establish two propositions.

(A) *If* (3) *has an* (L^{2p})*-solution then* $y'(x)\to 0$ *and* $y(x)\to 0$ *as* $x\to\infty$.

The proof of (A) follows that given for similar conclusions in [1].

(B) *The existence of an* (L^{2p})*-solution of* (3) *implies the existence of an* (L^2) *solution.*

Integrating (3) twice and using (A)

$$|y(x)| \leqq \int_x^\infty \left[\int_u^\infty |f(v)|\,|y(v)|^p dv\right] du \leqq \int_x^\infty u|f(u)|\,|y(u)|^p du.$$

Hence

$$\begin{aligned} \int_0^\infty |y(x)|^2 dx &\leqq \int_0^\infty \left[\int_x^\infty u|f(u)|\,|y(u)|^p du\right]^2 dx \\ &\leqq \int_0^\infty \left[\int_x^\infty u^2|f(u)|^2 du\right]\left[\int_x^\infty |y(u)|^{2p} du\right] dx \\ &\leqq \left[\int_0^\infty |y(x)|^{2p} dx\right]\left[\int_0^\infty x^3|f(x)|^2 dx\right]. \end{aligned} \tag{4}$$

Both integrals on the right converge, which establishes (B).

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Now suppose (2) holds and (3) has an (L^{2p})-solution for $p \geqq 1$. By (A) there exists a T such that $|y(t)| < 1$ for $t > T$; hence $|y(t)|^2 \geqq |y(t)|^{2p}$ for $p \geqq 1$ and $t > T$. Proceeding as in (4) with such a t as lower limit gives

$$\begin{aligned}\int_t^\infty |y(x)|^2 dx &\leqq \left[\int_t^\infty |y(x)|^{2p} dx\right]\left[\int_t^\infty x^3 |f(x)|^2 dx\right] \\ &\leqq \left[\int_t^\infty |y(x)|^2 dx\right]\left[\int_t^\infty x^3 |f(x)|^2 dx\right].\end{aligned}$$

Since $\int_t^\infty |y(x)|^2 dx > 0$, it follows that

$$\int_t^\infty x^3 |f(x)|^2 dx > 1$$

contradicting (2) and establishing the theorem.

Bibliography

1. A. Wintner, *A criterion for the non-existence of* (L^2)*-solutions of a nonoscillatory differential equation*, J. London Math. Soc. **25** (1950), 347–351.

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