A SPECTRAL MAPPING THEOREM FOR FUNCTIONS OF TWO COMMUTING LINEAR OPERATORS

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Let $\mathfrak{X}$ be a complex Banach space and $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$. Let $\mathfrak{A}$ and $\mathfrak{A}^*$ be the Banach algebras of all endomorphisms of $\mathfrak{X}$ and $\mathfrak{X}^*$, respectively; if $T \in \mathfrak{A}$ the adjoint $T^* \in \mathfrak{A}^*$. We use $\mathfrak{B}$ to denote the Banach algebra of all endomorphisms of $\mathfrak{A}$ considered as a Banach space. As in [3] we associate with any U, $V \in \mathfrak{A}$, the operators U^+, $V^- \in \mathfrak{B}$ defined by $U^+(X) = UX$ and $V^-(X) = XV$. Note that $U^+V^- = V^-U^+$. Let $f(\zeta_1, \zeta_2)$ be a single-valued function of two complex variables that is analytic in both variables in a domain that contains the product $\sigma(U^+) \times \sigma(V^-)$ of the spectra of the operators U^+ and V^-. Then $f(U^+, V^-)$ is given by (see J. Schwartz [4])

$$(1) \qquad \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} (\zeta_1 I^+ - U^+)^{-1}(\zeta_2 I^- - V^-)^{-1} f(\zeta_1, \zeta_2)\, d\zeta_1 d\zeta_2,$$

where Γ_1 and Γ_2 are suitable contours and I is the identity operator in $\mathfrak{A}$. The operator defined by (1) belongs to $\mathfrak{B}$.

In this note we determine the relationship among the spectra of $f(U^+, V^-)$, U^+ and V^- thus extending the well-known result of the case of a single variable. G. Lumer and M. Rosenblum [3] have paved the way for this result in their analysis covering the case $f(\zeta_1, \zeta_2) = \sum_{i=1}^{n} f_i(\zeta_1) g_i(\zeta_2)$. We also study point spectra and eigenvectors, and apply our results to the Fréchet derivative.

It is shown in [3] that $\sigma(U^+) = \sigma(U)$ and $\sigma(V^-) = \sigma(V)$. The relations for point spectra are given in the following lemma. Let $\sigma_p(T)$ denote the point spectrum of the operator T. For any $x \in \mathfrak{X}$ and $y^* \in \mathfrak{X}^*$ we define the operator $x \otimes y^* \in \mathfrak{A}$ by $(x \otimes y^*)z = y^*(z)x$.

LEMMA. *Suppose U, $V \in \mathfrak{A}$ and that $u \in \mathfrak{X}$ and $v^* \in \mathfrak{X}^*$ are eigenvectors of U and V^* with corresponding eigenvalues μ and ν, respectively. Then*

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[1] The author is indebted to the referee for pointing out that it was not essential to "work in finite dimension" as in the article originally submitted. The revision was written in the light of reference 3, brought to the author's attention by the referee, as well as references 1 and 4 which were discovered at about the same time. The author is presently with the National Marine Consultants Division of Interstate Electronics Corporation, Anaheim, California.

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(a) $U^+V^-(u\otimes v^*)=\mu\nu(u\otimes v^*)$,
(b) $\sigma_p(U^+)=\sigma_p(U)$,
(c) $\sigma_p(V^-)=\sigma_p(V^*)$.

PROOF. (a) follows from the fact that $U(u\otimes v^*)V=(Uu)\otimes(V^*v^*)$. Then (a) implies $\sigma_p(U)\subset\sigma_p(U^+)$ and $\sigma_p(V^*)\subset\sigma_p(V^-)$. Suppose that $\xi\in\sigma_p(U^+)$ and $\eta\in\sigma(V^-)$ with corresponding eigenvectors X and Y, respectively. Then there exist $a,\ b\in\mathfrak{X}$ such that $Xa\neq\theta,\ Yb\neq\theta$. Choose $y^*\in\mathfrak{X}^*$ with $y^*(Yb)=1$. By direct verification we see that $U(Xa)=\xi(Xa)$ and $V^*(Y^*y^*)=\eta(Y^*y^*)$. Hence $\xi\in\sigma_p(U)$ and $\eta\in\sigma(V^*)$. This completes the proof.

If $f(\zeta)$ is a single-valued function that is analytic in a complex domain that contains $\sigma(U)$, it is shown in [3] that

$$(f(U))^+ = f(U^+), \tag{2}$$

$$(f(U))^- = f(U^-). \tag{3}$$

THEOREM. *If* $f(U^+,\ V^-)$ *is defined by* (1) *then*

$$\sigma(f(U^+,\ V^-)) = f(\sigma(U),\ \sigma(V)). \tag{4}$$

Furthermore, if u *and* v^* *are eigenvectors of* U *and* V^* *with corresponding eigenvalues* μ *and* ν, *respectively, then* $u\otimes v^*$ *is an eigenvector of* $f(U^+,\ V^-)$ *corresponding to the eigenvalue* $f(\mu,\ \nu)$.

PROOF. Let $\lambda\in\sigma(f(U^+,\ V^-))$ and suppose $\lambda\notin f(\sigma(U),\ \sigma(V))$. Then the function $h(\zeta_1,\ \zeta_2)=(f(\zeta_1,\ \zeta_2)-\lambda)^{-1}$ is defined for $(U^+,\ V^-)$. Also, $h(U^+,\ V^-)(f(U^+,\ V^-)-\lambda I^+)=I^+$ which contradicts the assumption that $\lambda\in\sigma(f(U^+,\ V^-))$. The proof of (4) is completed in the manner of G. Lumer and M. Rosenblum [3, Theorems 9 and 10] on observing that for $X\in\mathfrak{A}$

$$f(U^+,\ V^-)X = \left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_1}\int_{\Gamma_2}(\zeta_1 I - U)^{-1}X(\zeta_2 I - V)^{-1}f(\zeta_1,\zeta_2)d\zeta_1 d\zeta_2.$$

The second part of the theorem may be verified directly with the aid of (2), (3) and the lemma.

Relation (4) is an extension of a result of Lumer and Rosenblum [3, Theorem 10]; M. Hausner [1] obtained (4) in the case of matrix algebras.

A function f with domain and range in $\mathfrak{A}$ is *analytic* at $X\in\mathfrak{A}$ if the Fréchet differential has the form

$$df(X,\ H) = g(X^+,\ X^-)H$$

where g has domain in $\mathfrak{B}\times\mathfrak{B}$ and range in $\mathfrak{B}$. As in [5] the derivative $f^1(X)=df(X,\ \cdot)=g(X^+,\ X^-)$. Let $f(\zeta)$ be a single-valued function of

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a complex variable that is analytic in a domain that contains $\sigma(X)$. Then $f(Z)$ is defined for $\|Z-X\|$ small and has a Fréchet derivative (see J. Schwartz [4])

$$f^1(X) = f(X^+, X^-)$$

where the right side is defined by (1) with

$$(5)\qquad \begin{aligned} f(\zeta_1, \zeta_2) &= \frac{f(\zeta_1) - f(\zeta_2)}{\zeta_1 - \zeta_2} && \text{if } \zeta_1 \neq \zeta_2 \\ &= f'(\zeta_1) && \text{if } \zeta_1 = \zeta_2. \end{aligned}$$

We then have the following corollary.

COROLLARY. *The spectrum of the Fréchet derivative considered as an operator on the Banach space* $\mathfrak{A}$ *is given by*

$$(6)\qquad \sigma(f^1(X)) = f(\sigma(X), \sigma(X))$$

where $f(\zeta_1, \zeta_2)$ *is defined by* (5). *If* x *and* y^* *are eigenvectors of* X *and* X^* *corresponding to eigenvalues* ξ *and* η, *respectively, then* $x \otimes y^*$ *is an eigenvector of* $f^1(X)$ *corresponding to the eigenvalue* $f(\xi, \eta)$.

M. Hausner [1] obtained (6) in the case of matrix algebras.

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