THE NECESSITY OF HARRIS' CONDITION FOR THE EXISTENCE OF A STATIONARY MEASURE

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1. **Introduction.** Let P be the matrix of transition probabilities for an irreducible transient Markov chain with state space the nonnegative integers. Harris obtains in [1] a sufficient condition for the existence of a positive vector solution to the equation

$$aP = a. \tag{1}$$

He feels that his condition is also close to being necessary, and the main purpose of this note is to prove this fact.[2] In §4 we present an unrelated result of some interest which we had mistakenly thought to be essential for Theorem 1.

2. **Facts about transient chains.** Recall that transience implies the sum

$$Q = \sum_{n=0}^{\infty} P^n \tag{2}$$

exists, while irreducibility guarantees to any pair of states i, j an $n>0$ with $P_{ij}^{(n)}>0$. The entries Q_{ik} of (2) of course represent the mean number of visits to k of a path which starts at i.

We introduce some notation which will simplify the proof in §4.

Let Ω be the set of nonnegative integer-valued sequences with coordinate functions $x(n, \omega)$. Assume $x(0, \omega) \equiv 0$. $\mathcal{B}$ will be the smallest Borel field which contains the cylinder sets

$$A(n_1, \cdots, n_k; r_1, \cdots, r_k) = \{\omega \mid x(n_j, \omega) = r_j; j = 1, \cdots, k\}.$$

The measure $\mathcal{P}$ will be the extension of the measure $\mathcal{P}'$ to $\mathcal{B}$, where $\mathcal{P}'$ is defined on cylinder sets by

$$\mathcal{P}'(A(n_1, \cdots, n_k; r_1, \cdots, r_k)) = P_{0r_1}^{(n_1)} P_{r_1r_2}^{(n_2-n_1)} \cdots P_{r_{k-1}r_k}^{(n_k-n_{k-1})}.$$

We recall for the reader's convenience a portion of the statement of the fundamental convergence theorem of Doob. (Cf. [2] for a complete exposition.)

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THEOREM. *Let* P, Q *be as in* §1. *For almost every* $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{Q_{i,x(n,\omega)}}{Q_{0,x(n,\omega)}} = q_i \tag{3}$$

exists and satisfies

$$\sum P_{ij} q_j = q_i. \tag{4}$$

This theorem will be necessary for the proof of Theorem 1.

3. **The necessary condition.** Let $L_{ki}(j)$ denote the probability that a path starting at k, reaches i, with the first visit preceded by a visit to a state with index $\geqq j$. $L_{ki}(0) = L_{ki}$ is the probability of ever reaching i from k. These quantities were introduced in [1] for $k \geqq j > i$.

THEOREM 1. *Suppose there exists a non-negative solution to* (1) *which is not identically* 0. *Then there is an infinite set* K *of states such that*

$$\lim_{j\to\infty, k\to\infty, k\in K} \frac{L_{ki}(j)}{L_{ki}} = 0, \qquad i = 0, 1, 2, \cdots. \tag{5}$$

PROOF. Fixing i, we will have for some $n > 0$, $P_{0i}^{(n)} > 0$. Since $L_{ki} \geqq L_{k0} P_{0i}^{(n)}$ and $Q_{k0} = L_{k0}(1 + Q_{00})$, the inequality

$$\frac{L_{ki}(j)}{L_{ki}} \leqq \frac{L_{ki}(j)}{Q_{k0}} \frac{1 + Q_{00}}{P_{0i}^{(n)}}$$

holds. It will therefore suffice to prove (5) for the ratios $L_{ki}(j)/Q_{k0}$.

Secondly, let

$${}_iP_{kj}^{(n)} = P_r\{x(n) = j;\, x(m) \neq i,\, 0 \leqq m < n \mid x(0) = k\}.$$

We could of course translate ${}_iP_{kj}^{(n)}$ into the notation introduced in §2, but this will not be necessary. What is essentially formula (4.2) in [1] is the easily checked

$$L_{ki}(j) = \sum_{r=j}^{\infty} \sum_{n=0}^{\infty} {}_iP_{kr}^{(n)} P_{ri}.$$

Obviously, ${}_iP_{kr}^{(n)} \leqq P_{kr}^{(n)}$, so

$$L_{ki}(j) \leqq \sum_{r=j}^{\infty} \sum_{n=0}^{\infty} P_{kr}^{(n)} P_{ri} = \sum_{r=j}^{\infty} Q_{kr} P_{ri}. \tag{6}$$

For $k \neq i$,

$$\sum_{r=0}^{\infty} Q_{kr} P_{ri} = Q_{ki}. \tag{7}$$

This well-known fact follows from the monotone convergence theorem applied to the partial sums for Q. From (6) and (7) we obtain for $k\neq i$,

$$L_{ki}(j) \leqq Q_{ki} - \sum_{r=0}^{j-1} Q_{kr}P_{ri}. \tag{8}$$

Equation (4.10) in [1] evaluates the deficiency in (8) which will be unnecessary for our purposes.

Thirdly, let $a=\{a_k\}$ be the solution to (1) in the statement of our theorem. Since a is not identically 0, and for all n, $aP^n=a$, irreducibility implies $a_k>0$ for all k. We form a reverse chain, also transient, governed by the matrix P^* below.

$$P^*_{ij} = \frac{a_j}{a_i} P_{ji}, \qquad Q^*_{ij} = \frac{a_j}{a_i} Q_{ji}. \tag{9}$$

Substituting equations (9) into equations (3) and (4), Doob's theorem guarantees that for almost every $\omega\in\Omega^*$

$$\liminf_{n\to\infty} \frac{Q_{x(n,\omega),i}}{Q_{x(n,\omega),0}} = s_i, \qquad i = 0, 1, 2, \cdots \tag{10}$$

exists and satisfies (1).

Fourthly, let K be the set consisting of the states visited by any sample path for which (10) is satisfied. Arranging K in increasing order will not affect the limit (10), due to the transience of the reverse chain.

Lastly, divide equation (8) by Q_{k0}. We will show

$$\lim_{j\to\infty, k\to\infty, k\in K} \left[\frac{Q_{ki}}{Q_{k0}} - \sum_{r=0}^{j-1} \frac{Q_{kr}}{Q_{k0}} P_{ri} \right] = 0. \tag{11}$$

By (7) the difference in (11) is always nonnegative. Thus if (11) is false, there will be an $\epsilon>0$ and a sequence (k_n, j_n), $k_n\in K$, $k_n, j_n\to\infty$ such that

$$\frac{Q_{k_n,i}}{Q_{k_n,0}} - \sum_{r=0}^{j_n-1} \frac{Q_{k_n,r}}{Q_{k_n,0}} P_{ri} \geqq \epsilon.$$

Let

$$f_n(r) = \begin{cases} \dfrac{Q_{k_n,r}}{Q_{k_n,0}}, & r < j_n, \\ 0, & r \geqq j_n, \end{cases} \tag{12}$$

be defined on the nonnegative integers Z^+. For $E \subset Z^+$ define the measure $\mu_i(\cdot)$ by

$$\mu_i(E) = \sum_{r \in E} P_{ri}.$$

By assumption,

$$\begin{aligned} \epsilon &\leqq \limsup_{n\to\infty} \left[f_n(i) - \int_{Z^+} f_n(r) d\mu_i(r) \right] \\ &\leqq \lim_{n\to\infty} f_n(i) - \liminf_{n\to\infty} \int_{Z^+} f_n(r) d\mu_i(r) \\ &\leqq \lim_{n\to\infty} f_n(i) - \int_{Z^+} \lim_{n\to\infty} f_n(r) d\mu_i(r) \\ &= 0. \end{aligned}$$

The last inequality follows from Fatou's lemma. The contradiction thus obtained and inequality (8) give us the desired result.

4. An unrelated result.

THEOREM 2. *Let P, Q be as in* §1 *and* $(\Omega, \mathcal{B}, \mathcal{P})$ *as in* §2. *For almost all* $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{P_{i,x(n,\omega)}}{Q_{i,x(n,\omega)}} = 0, \qquad i = 0, 1, 2, \cdots. \tag{13}$$

PROOF. The functions

$$g_{i,n}(\omega) = \frac{P_{i,x(n,\omega)}}{Q_{i,x(n,\omega)}}$$

are measurable, as are the functions

$$h_i(\omega) = \limsup_{n\to\infty} g_{i,n}(\omega)$$

Let A_i be the set of ω for which $h_i(\omega) > 0$. If Theorem 2 were false, we would have for some i, $\mathcal{P}(A_i) > 0$. Since $\mathcal{P}$ is countably additive, for N sufficiently large $\mathcal{P}(A_{i,N}) > 0$ where $A_{i,N}$ is the set where $h_i(\omega) > 1/N$. To each $\omega \in A_{i,N}$ associate the set

$$S(\omega) = \left\{ x(n\ \omega) \,\middle|\, \frac{P_{i,x(n,\omega)}}{Q_{i,x(n,\omega)}} > \frac{1}{N} \right\}.$$

Note that each path ω visits $S(\omega)$ infinitely often. Let

$$K = \bigcup_{\omega \in A_{i,N}} S(\omega).$$

For each $k \in K$,

$$\frac{P_{ik}}{Q_{ik}} > \frac{1}{N} \cdot$$

This implies

$$\sum_{k \in K} Q_{ik} < N \sum_{k \in K} P_{ik} \leqq N. \tag{14}$$

The sum on the left represents the mean number of times a path from i visits the set K. There is a positive probability of a path from i visiting 0 so the sum (14) for $i=0$ must also be finite. But each $\omega \in A_{i,N}$ visits K infinitely often, and the assumption that $A_{i,N}$ has positive measure gives us a contradiction, proving the theorem.

From the proof follows the

COROLLARY. *For almost all $\omega \in \Omega$, and all i, j, k*

$$\lim_{n \to \infty} \frac{P^{(k)}_{i,x(n,\omega)}}{Q_{j,x(n,\omega)}} = 0.$$

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