

# THE NECESSITY OF HARRIS' CONDITION FOR THE EXISTENCE OF A STATIONARY MEASURE

WILLIAM VEECH<sup>1</sup>

**1. Introduction.** Let  $P$  be the matrix of transition probabilities for an irreducible transient Markov chain with state space the nonnegative integers. Harris obtains in [1] a sufficient condition for the existence of a positive vector solution to the equation

$$(1) \quad aP = a.$$

He feels that his condition is also close to being necessary, and the main purpose of this note is to prove this fact.<sup>2</sup> In §4 we present an unrelated result of some interest which we had mistakenly thought to be essential for Theorem 1.

**2. Facts about transient chains.** Recall that transience implies the sum

$$(2) \quad Q = \sum_{n=0}^{\infty} P^n$$

exists, while irreducibility guarantees to any pair of states  $i, j$  an  $n > 0$  with  $P_{ij}^{(n)} > 0$ . The entries  $Q_{ik}$  of (2) of course represent the mean number of visits to  $k$  of a path which starts at  $i$ .

We introduce some notation which will simplify the proof in §4.

Let  $\Omega$  be the set of nonnegative integer-valued sequences with coordinate functions  $x(n, \omega)$ . Assume  $x(0, \omega) \equiv 0$ .  $\mathfrak{B}$  will be the smallest Borel field which contains the cylinder sets

$$A(n_1, \dots, n_k; r_1, \dots, r_k) = \{\omega \mid x(n_j, \omega) = r_j; j = 1, \dots, k\}.$$

The measure  $\mathcal{P}$  will be the extension of the measure  $\mathcal{P}'$  to  $\mathfrak{B}$ , where  $\mathcal{P}'$  is defined on cylinder sets by

$$\mathcal{P}'(A(n_1, \dots, n_k; r_1, \dots, r_k)) = P_{0r_1}^{(n_1)} P_{r_1 r_2}^{(n_2 - n_1)} \dots P_{r_{k-1} r_k}^{(n_k - n_{k-1})}.$$

We recall for the reader's convenience a portion of the statement of the fundamental convergence theorem of Doob. (Cf. [2] for a complete exposition.)

Received by the editors September 11, 1962.

<sup>1</sup> This research is sponsored by the United States Air Force under project RAND-Contract No. AF 49(638)-700-monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Technology. Hq USAF.

<sup>2</sup> The author wishes to thank T. E. Harris for posing this problem.

**THEOREM.** *Let  $P, Q$  be as in §1. For almost every  $\omega \in \Omega$ ,*

$$(3) \quad \lim_{n \rightarrow \infty} \frac{Q_{i,x(n,\omega)}}{Q_{0,x(n,\omega)}} = q_i$$

*exists and satisfies*

$$(4) \quad \sum P_{ij}q_j = q_i.$$

This theorem will be necessary for the proof of Theorem 1.

**3. The necessary condition.** Let  $L_{ki}(j)$  denote the probability that a path starting at  $k$ , reaches  $i$ , with the first visit preceded by a visit to a state with index  $\geq j$ .  $L_{ki}(0) = L_{ki}$  is the probability of ever reaching  $i$  from  $k$ . These quantities were introduced in [1] for  $k \geq j > i$ .

**THEOREM 1.** *Suppose there exists a non-negative solution to (1) which is not identically 0. Then there is an infinite set  $K$  of states such that*

$$(5) \quad \lim_{j \rightarrow \infty, k \rightarrow \infty, k \in K} \frac{L_{ki}(j)}{L_{ki}} = 0, \quad i = 0, 1, 2, \dots$$

**PROOF.** Fixing  $i$ , we will have for some  $n > 0$ ,  $P_{0i}^{(n)} > 0$ . Since  $L_{ki} \geq L_{k0}P_{0i}^{(n)}$  and  $Q_{k0} = L_{k0}(1 + Q_{00})$ , the inequality

$$\frac{L_{ki}(j)}{L_{ki}} \leq \frac{L_{ki}(j)}{Q_{k0}} \frac{1 + Q_{00}}{P_{0i}^{(n)}}$$

holds. It will therefore suffice to prove (5) for the ratios  $L_{ki}(j)/Q_{k0}$ .

Secondly, let

$${}_iP_{kj}^{(n)} = P_r \{ x(n) = j; x(m) \neq i, 0 \leq m < n \mid x(0) = k \}.$$

We could of course translate  ${}_iP_{kj}^{(n)}$  into the notation introduced in §2, but this will not be necessary. What is essentially formula (4.2) in [1] is the easily checked

$$L_{ki}(j) = \sum_{r=j}^{\infty} \sum_{n=0}^{\infty} {}_iP_{kr}^{(n)} P_{ri}.$$

Obviously,  ${}_iP_{kr}^{(n)} \leq P_{kr}^{(n)}$ , so

$$(6) \quad L_{ki}(j) \leq \sum_{r=j}^{\infty} \sum_{n=0}^{\infty} P_{kr}^{(n)} P_{ri} = \sum_{r=j}^{\infty} Q_{kr}P_{ri}.$$

For  $k \neq i$ ,

$$(7) \quad \sum_{r=0}^{\infty} Q_{kr}P_{ri} = Q_{ki}.$$

This well-known fact follows from the monotone convergence theorem applied to the partial sums for  $Q$ . From (6) and (7) we obtain for  $k \neq i$ ,

$$(8) \quad L_{ki}(j) \leq Q_{ki} - \sum_{r=0}^{j-1} Q_{kr}P_{ri}.$$

Equation (4.10) in [1] evaluates the deficiency in (8) which will be unnecessary for our purposes.

Thirdly, let  $a = \{a_k\}$  be the solution to (1) in the statement of our theorem. Since  $a$  is not identically 0, and for all  $n$ ,  $aP^n = a$ , irreducibility implies  $a_k > 0$  for all  $k$ . We form a reverse chain, also transient, governed by the matrix  $P^*$  below.

$$(9) \quad P_{ij}^* = \frac{a_j}{a_i} P_{ji}, \quad Q_{ij}^* = \frac{a_j}{a_i} Q_{ji}.$$

Substituting equations (9) into equations (3) and (4), Doob's theorem guarantees that for almost every  $\omega \in \Omega^*$

$$(10) \quad \liminf_{n \rightarrow \infty} \frac{Q_{z(n,\omega),i}}{Q_{z(n,\omega),0}} = s_i, \quad i = 0, 1, 2, \dots$$

exists and satisfies (1).

Fourthly, let  $K$  be the set consisting of the states visited by any sample path for which (10) is satisfied. Arranging  $K$  in increasing order will not affect the limit (10), due to the transience of the reverse chain.

Lastly, divide equation (8) by  $Q_{k0}$ . We will show

$$(11) \quad \lim_{j \rightarrow \infty, k \rightarrow \infty, k \in K} \left[ \frac{Q_{ki}}{Q_{k0}} - \sum_{r=0}^{j-1} \frac{Q_{kr}}{Q_{k0}} P_{ri} \right] = 0.$$

By (7) the difference in (11) is always nonnegative. Thus if (11) is false, there will be an  $\epsilon > 0$  and a sequence  $(k_n, j_n)$ ,  $k_n \in K$ ,  $k_n, j_n \rightarrow \infty$  such that

$$\frac{Q_{k_n,i}}{Q_{k_n,0}} - \sum_{r=0}^{j_n-1} \frac{Q_{k_n,r}}{Q_{k_n,0}} P_{ri} \geq \epsilon.$$

Let

$$(12) \quad f_n(r) = \begin{cases} \frac{Q_{k_n,r}}{Q_{k_n,0}}, & r < j_n, \\ 0, & r \geq j_n, \end{cases}$$

be defined on the nonnegative integers  $Z^+$ . For  $E \subset Z^+$  define the measure  $\mu_i(\cdot)$  by

$$\mu_i(E) = \sum_{r \in E} P_{ri}.$$

By assumption,

$$\begin{aligned} \epsilon &\leq \limsup_{n \rightarrow \infty} \left[ f_n(i) - \int_{Z^+} f_n(r) d\mu_i(r) \right] \\ &\leq \lim_{n \rightarrow \infty} f_n(i) - \liminf_{n \rightarrow \infty} \int_{Z^+} f_n(r) d\mu_i(r) \\ &\leq \lim_{n \rightarrow \infty} f_n(i) - \int_{Z^+} \lim_{n \rightarrow \infty} f_n(r) d\mu_i(r) \\ &= 0. \end{aligned}$$

The last inequality follows from Fatou's lemma. The contradiction thus obtained and inequality (8) give us the desired result.

#### 4. An unrelated result.

**THEOREM 2.** *Let  $P, Q$  be as in §1 and  $(\Omega, \mathfrak{B}, \mathfrak{P})$  as in §2. For almost all  $\omega \in \Omega$ ,*

$$(13) \quad \lim_{n \rightarrow \infty} \frac{P_{i,x(n,\omega)}}{Q_{i,x(n,\omega)}} = 0, \quad i = 0, 1, 2, \dots$$

**PROOF.** The functions

$$g_{i,n}(\omega) = \frac{P_{i,x(n,\omega)}}{Q_{i,x(n,\omega)}}$$

are measurable, as are the functions

$$h_i(\omega) = \limsup_{n \rightarrow \infty} g_{i,n}(\omega)$$

Let  $A_i$  be the set of  $\omega$  for which  $h_i(\omega) > 0$ . If Theorem 2 were false, we would have for some  $i$ ,  $\mathfrak{P}(A_i) > 0$ . Since  $\mathfrak{P}$  is countably additive, for  $N$  sufficiently large  $\mathfrak{P}(A_{i,N}) > 0$  where  $A_{i,N}$  is the set where  $h_i(\omega) > 1/N$ . To each  $\omega \in A_{i,N}$  associate the set

$$S(\omega) = \left\{ x(n, \omega) \mid \frac{P_{i,x(n,\omega)}}{Q_{i,x(n,\omega)}} > \frac{1}{N} \right\}.$$

Note that each path  $\omega$  visits  $S(\omega)$  infinitely often. Let

$$K = \bigcup_{\omega \in A_{i,N}} S(\omega).$$

For each  $k \in K$ ,

$$\frac{P_{ik}}{Q_{ik}} > \frac{1}{N}.$$

This implies

$$(14) \quad \sum_{k \in K} Q_{ik} < N \sum_{k \in K} P_{ik} \leq N.$$

The sum on the left represents the mean number of times a path from  $i$  visits the set  $K$ . There is a positive probability of a path from  $i$  visiting 0 so the sum (14) for  $i=0$  must also be finite. But each  $\omega \in A_{i,N}$  visits  $K$  infinitely often, and the assumption that  $A_{i,N}$  has positive measure gives us a contradiction, proving the theorem.

From the proof follows the

COROLLARY. *For almost all  $\omega \in \Omega$ , and all  $i, j, k$*

$$\lim_{n \rightarrow \infty} \frac{P_{i,x(n,\omega)}^{(k)}}{Q_{j,x(n,\omega)}} = 0.$$

#### REFERENCES

1. T. E. Harris, *Transient Markov chains with stationary measures*, Proc. Amer. Math. Soc. 8 (1957), 937-942.
2. G. A. Hunt, *Markoff chains and Martin boundaries*, Illinois J. Math. 4 (1960), 313-340.

THE RAND CORPORATION